Convergence of a streamline method for hyperbolic problems

Bilal ATFEH
CMI, 39, rue F. Joliot Curie, 13453 Marseille Cedex 13
atfeh@cmi.univ-mrs.fr

Abstract

In this paper we study the convergence of a streamline method for an hyperbolic problem. Our motivation for this method arises from the problem of simulating multi-phase flow in porous media.

In fact, the streamline method has been applied successfully to reservoir simulation. There is however no study of the convergence of this method. We prove the convergence in a simplified case. In particular, we assume that the velocity depends only on the space variable.

Key words : Streamline method, finite volume method.

Contents

1 Introduction

In this work, we study the convergence of the streamline method applied to the following hyperbolic problem:

$$\partial_t u + \mathrm{div}(f(u)\overrightarrow{V}) = 0,$$

where f is an increasing function.
Our motivation for using this method arises from the problem of modelling multi-phase flow in porous media. In fact, u can be interpreted as the saturation of one phase (water, oil or gaz) and then $\overrightarrow{V}$ is the total velocity which depends on time, saturation and the pressure P. Usually $\overrightarrow{V}$ is computed with Darcy's law, and satisfies the following elliptic problem:

$$\mathrm{div}(\overrightarrow{V}(u,t,P)) = 0.$$

Flow simulation in porous media is often studied using either a finite difference method ([3]), a finite element method ([9]) or a finite volume method ([11], [12],[14]). Recent studies have however shown the efficiency of the streamline method, in particular in the reduction of the computing time, see [1], [5], [6] and [7].

The original idea of this method consists in tracing lines in the domain (streamlines compared to the total velocity) then to transform the equation satisfied by u into a 1-d equation along these curves, using only the fact that $\mathrm{div}(\overrightarrow{V}) = 0$ (see [5], [2]).

Numerical tests have showed a possible extension of this method to basin modeling [1], even in this case $\mathrm{div}(\overrightarrow{V}) \neq 0$, because of compaction (see [17] and [18]).

Concerning the mathematical analysis, there are no results on this method such as convergence, stability. This work thus constitutes a first step towards a more complete study of this method.

The model we treat here is simpler than the physical model, the main differences can be sumed up in the following points:

1. The velocity depends only on the space variable x. We however put no condition on $\mathrm{div}(\overrightarrow{V})$.
2. The velocity $\overrightarrow{V}$ is supposed to be regular as well as the solution u of the problem.

1.1 Problem setting

Let Ω be an open polyhedral bounded subset of $\mathbb{R}^d$ ($d = 2$ or 3), denote by $\partial\Omega$ its boundary.
Let $I = (0,T)$ be an interval of $\mathbb{R}_+$ ($T > 0$).
Let $Q = \Omega \times I$, $\Sigma = \partial\Omega \times I$, $\overline{Q} = \overline{\Omega} \times I$ and $\Sigma_- = \left\{(x,t) \in \Sigma : \overrightarrow{V}(x).\overrightarrow{n}(x) \leq 0\right\}$, and $\overrightarrow{n}$ be the outward unit normal accross $\partial\Omega$.
We consider the following problem:

$$(\mathcal{P}) = \begin{cases} \partial_t u(x,t) + \mathrm{div}(f(u(x,t))\overrightarrow{V}(x)) = 0 & , \quad \forall (x,t) \in Q \\ u(x,0) = u_0(x) & , \quad \forall x \in \Omega \\ u(r,t) = g(r,t) & , \quad \forall (r,t) \in \Sigma_-. \end{cases}$$

In $(\mathcal{P})$, the scalar valued function u is the unknown, whereas the other quantities are given data of the problem. They are assumed to satisfy the following hypotheses (D):

(D1) $f \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$, satisfies: $f' > 0$.

(D2) $\overrightarrow{V} \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d)$, satisfies:

$$\operatorname{div}(\overrightarrow{V}) \in \mathcal{C}^1(\Omega, \mathbb{R}), \tag{1}$$

$$\exists \eta \in \mathbb{R}_+^* \ , \forall x \in \overline{\Omega},\ \eta \leq |\overrightarrow{V}(x)| \tag{2}$$

one sets $\beta = |||\overrightarrow{V}|||_{L^\infty(\Omega)}$ ($\beta < \infty$ since Ω is bounded).

(D3) The functions u_0 and g are given data and are supposed to be rather regular (of class $\mathcal{C}^2$).

To study this kind of problem, one introduces the concept of **entropy solution**, which enables a uniqueness result ([8], [13]). Indeed the entropy solution represents the physical solution among weak solutions. We now recall the definition of weak entropy solution.

Definition 1.1 *Weak entropy solution*
Let $u \in L^\infty(Q)$. The function u is said to be a weak entropy solution to the problem $(\mathcal{P})$ if it satisfies the following entropy inequalities: for all $\kappa \in \mathbb{R}$, and for all $\phi \in \mathcal{D}$, $\phi \geq 0$,

$$\begin{aligned} &\int_Q (|u(x,t) - \kappa| \partial_t \phi(x,t) + |f(u(x,t)) - f(\kappa)| \overrightarrow{V}(x).\overrightarrow{\nabla}\phi(x,t)) dx\ dt \\ &+ \int_\Omega |u_0(x) - \kappa| \phi(x,0) dx - \int_\Sigma |f(g(r,t)) - f(\kappa)| \phi(r,t) \overrightarrow{V}(r).\overrightarrow{n}(r) d\gamma(r) dt \quad \geq 0. \end{aligned} \tag{3}$$

Where $\mathcal{D} = \{\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}_+, \mathbb{R}), \phi = 0$ on $\Sigma - \Sigma_-\}$, and $d\gamma$ is the $(d-1)$ Lebesgue measure on $\partial\Omega$.

With this definition, we have the following result:

Theorem 1.2 *([4]) There exists a unique weak entropy solution $u \in L^1 \cap BV(Q)$ to the problem $(\mathcal{P})$.*

Remark 1 *The definition of the weak entropy solution given above accounts for the fact that the function f is increasing. For the general case the reader can refer to ([19]).*

In the following, we suppose that $u \in \mathcal{C}^2(\overline{Q})$. Indeed, in order to determine the equation verified by u on the streamline, it is essential for u to be of class $\mathcal{C}^1$. Furthermore, we take u of class $\mathcal{C}^2$ in order to have an error estimate of order h (see part 5.1).

1.2 Definitions: streamline, time of flight

For the convenience of the reader, we recall the following definitions:

Definition 1.3 *Streamlines are integral curves of the vector field $\overrightarrow{V}$, defined as tangent to $\overrightarrow{V}$ at each point.*

A streamline $\mathcal{C}$ can be interpreted as the trajectory of a particle moving along $\mathcal{C}$ with a velocity which, at every poiny p of $\mathcal{C}$, equals $\overrightarrow{V}(p)$.

Definition 1.4 *Time of flight (TOF)*
Let $\mathcal{C}$ be a streamline, and let us fix a point $p_0 \in \mathcal{C}$. For all points p of $\mathcal{C}$, the TOF between p_0 and p, denoted by $\tau(p_0, p)$, is the time required for a particle (which is moving along $\mathcal{C}$, with a velocity equal to $\overrightarrow{V}$) to go from p_0 to p. Furthermore, we can show that $\tau(p_0, p)$ is unique (as a solution of a Cauchy problem).

Based on the definition of the TOF, we can define a parametrization q of $\mathcal{C}$, depending on $\overrightarrow{V}$ and p_0, as follow:

$$\left\{ \begin{array}{lcl} q : I_{\mathcal{C}} \subset \mathbb{R} & \longrightarrow & \overline{\Omega} \\ \\ \qquad\qquad \tau & \longrightarrow & p = q(\tau), \quad \text{with: } \tau(p_0, p) = \tau. \end{array} \right. \tag{4}$$

In fact, q solves the following Cauchy problem: find $(I_{\mathcal{C}} \subset \mathbb{R}, q : I_{\mathcal{C}} \rightarrow \overline{\Omega})$, maximal solution of:

$$\left\{ \begin{array}{rcl} \dfrac{\overrightarrow{dq}}{d\tau}(\tau) & = & \overrightarrow{V}(q(\tau)), \\ \\ q(0) & = & p_0. \end{array} \right. \tag{5}$$

Since $\overrightarrow{V} \in \mathcal{C}^2(\Omega)$, we know that (5) admits a unique maximal solution, furthermore $q \in \mathcal{C}^3(I_{\mathcal{C}}, \Omega)$.

In the following, we call p_0 the **initial point** of $\mathcal{C}$, and $I_{\mathcal{C}}$ the **interval of definition** of $\mathcal{C}$, and each streamline will be characterized by its initial point, its interval of definition and q.

1.3 Equation satisfied by u on a streamline

Let $\mathcal{C}$ be a streamline traced in $\overline{\Omega}$, and $p \in \overline{\Omega}$ its initial point. Supposing that $\mathcal{C} = \{q(\tau) : \tau \in I_{\mathcal{C}} \subset \mathbb{R}\}$.
We define a new function $u_{\mathcal{C}}$ by:

$$u_{\mathcal{C}}(\tau, t) = u(q(\tau), t) \ , \ \ \forall (\tau, t) \in I_{\mathcal{C}} \times I. \tag{6}$$

The derivative of $u_{\mathcal{C}}$ with respect to t is given by

$$\partial_t u_{\mathcal{C}}(\tau, t) = \partial_t u(q(\tau), t)).$$

By derivating $u_{\mathcal{C}}$ with respect to τ, we obtain

$$\partial_\tau u_{\mathcal{C}}(\tau, t)) = \overrightarrow{\nabla} u(q(\tau), t) \cdot \overrightarrow{\partial_\tau q}(\tau) = \overrightarrow{\nabla} u.\overrightarrow{V}(q(\tau), t).$$

These relations yield:

$$\begin{array}{rcl} \partial_\tau f(u_{\mathcal{C}}(\tau, t)) & = & f'(u_{\mathcal{C}}(\tau, t))\overrightarrow{\nabla} u \cdot \overrightarrow{V}(q(\tau), t) \\ & = & \operatorname{div}\Big(f(u(q(\tau), t))\overrightarrow{V}(q(\tau))\Big) - f(u_{\mathcal{C}}(\tau, t))\operatorname{div}(\overrightarrow{V}(q(\tau)). \end{array}$$

Since u is the solution to the problem $(\mathcal{P})$, then $u_{\mathcal{C}}$ solves the following problem

$$\partial_t u_{\mathcal{C}}(\tau, t) + \partial_\tau f(u_{\mathcal{C}}(\tau, t)) + f(u_{\mathcal{C}}(\tau, t))\operatorname{div}(\overrightarrow{V}(q(\tau))) = 0, \ \forall (\tau, t) \in I_{\mathcal{C}} \times I. \tag{7}$$

Remark 2 *If $div(\overrightarrow{V}) = 0$ on Ω, then $u_{\mathcal{C}}$ solves a 1-d equation along the streamline $\mathcal{C}$.*

Solving equation (7) along each streamline is the so-called streamline method.

2 Discrete problem

This section is devoted to introduces the mesh notations that will be used in the sequel

2.1 The mesh on Ω

Let $\mathcal{M}$ be a mesh over Ω. For all control volume K of $\mathcal{M}$, we denote by:

1. $m(K)$ the d-dimensional measure of K,
2. $\mathcal{N}(K)$ the set of control volumes neighbour of K in $\mathcal{M}$,
3. σ_{KL} the common edge between K and $L \in \mathcal{N}(K)$,
4. $\mathcal{A}$ the set of edges of $\mathcal{M}$,
$$\mathcal{A} = \{\sigma_{KL},\ K \in \mathcal{M},\ L \in \mathcal{N}(L)\}, \tag{8}$$
$\mathcal{A}_\partial$ the set of the edges included in $\partial\Omega$, and $\mathcal{A}_\partial^-$ the set of edges included in Σ_-.
5. $m(\sigma)$ the $(d-1)$-dimensional Lebesgue measure of the edge σ.

We suppose that $\mathcal{M}$ satisfies:

- For two distinct control volumes K and L in $\mathcal{M}$, either $m(\overline{K} \cap \overline{L}) = 0$, or $\overline{K} \cap \overline{L} = \overline{\sigma}$ for some $\sigma \in \mathcal{A}$.
- For each $\sigma \in \mathcal{A}$, σ is contained in a hyperplane of $\mathbb{R}^d$.
- $\overline{\partial\Omega} = \cup_{\sigma \in \mathcal{A}_\partial} \overline{\sigma}$.
- If $\sigma \in \mathcal{A}_\partial$, then either $\overline{\sigma} \subset \mathcal{A}_\partial^-$, or $\overline{\sigma} \subset \mathcal{A}_\partial - \mathcal{A}_\partial^-$.

We also define the size of the mesh by $h = \sup\limits_{K \in \mathcal{M}} \{\text{diam}(K)\}$. Using the definition of h one has
$$\forall K \in \mathcal{M}\ ,\ m(K) \leq Ch^d, \tag{9}$$
with $C = 2^d m(B(0,1))$, $B(0,1)$ the unit ball of $\mathbb{R}^d$.
A mesh satisfing these properties is called a **regular mesh** on Ω.

2.2 Local mesh on a streamline

In this subsection, we assume $\mathcal{M}$ to be a regular mesh over Ω.
Let $N_\mathcal{L} \in \mathbb{N}^*$. Given $N_\mathcal{L}$ points $p^l \in \partial\Omega_-$, (*i.e.* $\overrightarrow{V}.\overrightarrow{n}(p^l) \leq 0$), from each point p^l we trace a streamline l. We denote by $\mathcal{L}$ the set of all streamlines traced in Ω.
For each $l \in \mathcal{L}$, l will be characterized by a triplet (p^l, I_l, q^l), where:

1. $p^l \in \partial\Omega$ is the initial point of l
2. I_l is of finite measure in $\mathbb{R}$. More preciesely, we suppose that:
there exists a constant $C_{\Omega,V} \in \mathbb{R}_+^*$ depending on Ω and $\overrightarrow{V}$, such that
$$\forall l \in \mathcal{L},\ \ |I_l| < C_{\Omega,V}. \tag{10}$$
3. $q^l : I_l = [0, \tau^l_{max}] \to \Omega$ is a solution of:
$$\begin{cases} \overrightarrow{\dfrac{dq^l}{d\tau}}(\tau) &=& \overrightarrow{V}(q^l(\tau))\ , \quad \forall \tau \in I_l, \\ q^l(0) &=& p^l. \end{cases}$$

A mesh over a streamline will indicate a mesh over its interval of definition, more precisely, for each $l \in \mathcal{L}$, a subdivision $\Gamma^l = (\tau_i^l)_{0\le i\le M_l}$ of I_l is given, and one denotes:

$$\begin{cases} \Delta_i^l &= \tau_{i+1}^l - \tau_i^l,\ 0 \le i \le M_l - 1, \\ \Delta^l &= \max\{\Delta_i^l,\ 0 \le i \le M_l - 1\}. \end{cases}$$

Δ^l is the size of the mesh over l. We suppose that the given mesh on l satisfies the following condition:

$$\exists \alpha^l \in [0,1),\ \forall 0 \le i < M_l - 1 :\ \alpha^l \Delta^l \le \Delta_i^l. \tag{11}$$

We denote also by $q_i^l = q(\tau_i^l)$ and h_i^l the length of $\widehat{q_i^l q_{i+1}^l}$ on l, given by:

$$h_i^l = \int_{[\tau_i^l, \tau_{i+1}^l)} |\overrightarrow{V}(q^l(\tau))| d\tau.$$

It is clear that $h_i^l \le \Delta_i^l \beta$. One also notes that $h_l = \sup_j (h_l^j)$.

We also introduce the following notations which we shall use to go from the mesh $\mathcal{M}$ to the set of streamlines.
Let $(K, l) \in \mathcal{M} \times \mathcal{L}$. We denote by:

- $I_K = \{l \in \mathcal{L},\ l \cap K \ne \emptyset\}$, the set of streamlines which meet K (see Fig-1).
- $J_l = \{K \in \mathcal{M},\ l \in I_K\}$, the set of control volumes which meet l.
- $I_{K,l} = \big\{0 \le j \le M_l - 1,\ \{q^l(\tau), \tau \in [\tau_j^l; \tau_{j+1}^l)\} \subseteq K\big\}$ (c.f. remark 3 below).

For each control volume K, we define the notion of "time of flight" (α_K) in K, as follow:

$$\alpha_K = \sum_{l \in I_K} \Delta_K^l,$$

where

$$\Delta_K^l = \sum_{j \in I_{K,l}} \Delta_j^l.$$

The following assumptions are made (HML):

(HML1) For each control volume $K \in \mathcal{M}$: $\alpha_K > 0$, in other words

$$\forall K \in \mathcal{M}, \qquad I_K \ne \emptyset.$$

Therefore, each control volume contains part of a streamline.

(HML2) $\exists\ 0 < c_{inf} \le c_{max} < \infty$, such that:

$$\forall l \in \mathcal{L},\ \forall j \in \{0, ..., M_l - 1\} :\ c_{inf} h \le \Delta_j^l \le \Delta^l \le c_{max} h.$$

(HML3) For $h < 1$, we suppose that: $\exists c_{NLDC} \in \mathbb{R}_+^*$ such that :

$$N_{\mathcal{L}} \le c_{NLDC}/h^{d-1},$$

in general $N_{\mathcal{L}}$ is proportional to the number of edges in $\mathcal{A}_\partial^-$.

(HML4) For each streamline $l \in \mathcal{L}$, $(l \cap \mathcal{A}) \subseteq (q_i^l)_i$.

Remark 3 • *The assumption* (HML1) *is easy to realize. In fact, we define* $H = \{K \in \mathcal{M},\, \alpha_K = 0\} = \{K \in \mathcal{M}, K \cap \mathcal{L} = \emptyset\}$. *For all* $K \in H$ *we trace a streamline* l^K *such that:* $p^{l^K} \in K$, *and now define*

$$\mathcal{L}^1 = \mathcal{L} \bigcup (\cup_{K \in H} l^K),$$

so that (HML1) *is satisfied with the new set of streamlines* $\mathcal{L}^1$.

- *The assumption* (HML4) *at the points of intersection between the streamlines and the set of the edges of the mesh* $\mathcal{M}$ *gives us:*

$$\forall (l, j) \in \mathcal{L} \times \{0, ..., M_l - 1\},\ \exists! K \in \mathcal{M} : \{q^l(\tau), \tau \in [\tau^l_j, \tau^l_{j+1})\} \subseteq K.$$

Furthermore, in numerical computation this assumption is checked automatically (see [10], [16], [15] and [1]).

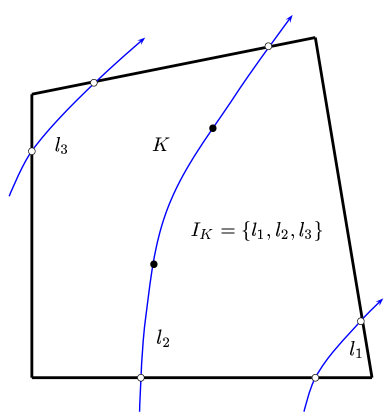

Fig 1. *Streamline path throught grid cell K*

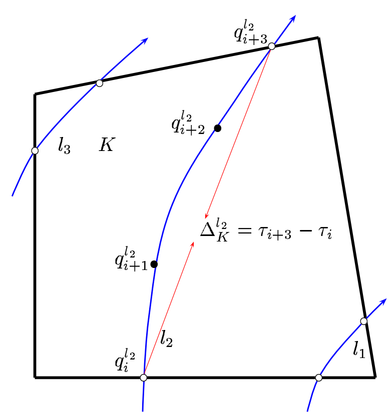

Fig 2. *Numerical computation of* Δ^l_K

3 Mapping between streamlines and mesh $\mathcal{M}$

In this part we give definitions of spaces which will be used in the following. $\mathcal{E}_\mathcal{M}$ will be the space of the functions defined on Ω, and for each streamline l, $\mathcal{E}_l$ will be the space of the functions defined on this line. To facilitate the computation of error estimates, we introduce the space $\mathcal{E}_\mathcal{L}$ which is the product of spaces $\mathcal{E}_l$.

3.1 Definition of functional spaces $\mathcal{E}_l$, $\mathcal{E}_\mathcal{L}$ et $\mathcal{E}_\mathcal{M}$

For each $l \in \mathcal{L}$, we define $\mathcal{E}_l$ as the set of piecewise constant functions on l with respect to the mesh given on l:

$$\mathcal{E}_l = \{\varphi : l \to \mathbb{R}, \text{such that} : \exists (\varphi_j)_{0 \le j < M_l} \in \mathbb{R}^{M_l} : \varphi \circ q^l(\tau) = \sum_{j=0}^{M_l - 1} \varphi_j 1_{[\tau^l_j, \tau^l_{j+1})}(\tau)\}.$$

Where $1_{[\tau^l_j, \tau^l_{j+1})}(\tau) = 1$ if $\tau \in [\tau^l_j, \tau^l_{j+1})$ and 0 elsewhere.
For each $\varphi \in \mathcal{E}_l$, we define:

$$\|\varphi\|_l = \sum_{j=0}^{M_l - 1} \Delta^l_j |\varphi_j| \tag{12}$$

This defines a norm on $\mathcal{E}_l$ which can be compared with the L^1 norm on l.

Remark 4 *The choice of Δ_j^l in the definition of the norm instead of h_j is made for two reasons:*

1. *As the velocity is regular, the two definitions are mathematically equivalent (c.f. lemma 3.1 afterwards)*
2. *In numerical computation it is much easier to evaluate Δ_j^l than h_j (see [1], [5] and [15]).*

Lemma 3.1 *Let $\varphi \in \mathcal{E}_l$. Then*

$$\|\varphi\|_{L^1(l)} \leq \beta \|\varphi\|_l. \tag{13}$$

Recall that β is defined by $\beta = \||\vec{V}|\|_{L^\infty(\Omega)}$.

Proof: On one hand, by definition of $\mathcal{E}_l$, there exists $(\varphi_j)_{0\leq j<M_l}$ such that:

$$\forall \tau \in I_l : \ \varphi \circ q^l(\tau) = \sum_j \varphi_j 1_{[\tau_j^l, \tau_{j+1}^l)}(\tau).$$

One thus has

$$\begin{aligned}\int_l |\varphi| \, dl &= \int_{I_l} |\varphi \circ q^l(\tau)| \|\vec{V}(q^l(\tau)\| \, d\tau \\ &= \sum_{j=0}^{M_l-1} |\varphi_j| h_j^l\end{aligned}$$

On the other hand, one has:

$$\forall l \in \mathcal{L}; \ \forall j \in \{0, ..., M_l - 1\} \ : h_j^l \leq \beta \Delta_j^l,$$

and the result follows. ∎

Remark 5 *We can identify $\mathcal{E}_l$ with the piecewise constant functions on I_l. In fact both spaces are real vector spaces of dimension equal to M_l.*

We also define on the set of all streamlines $\mathcal{L}$ the following space:

$$\mathcal{E}_\mathcal{L} = \{\Phi = (\varphi^l)_{0\leq l\leq N_\mathcal{L}}, \varphi^l \in \mathcal{E}_l\}.$$

We provide $\mathcal{E}_\mathcal{L}$ with the norm $\|.\|_\mathcal{L}$, defined by

$$\forall \Phi = (\varphi^l) \in \mathcal{E}_\mathcal{L}, \|\Phi\|_\mathcal{L} = \sum_{l\in\mathcal{L}} \|\varphi^l\|_l. \tag{14}$$

We also define on Ω the set $\mathcal{E}_\mathcal{M}$ of piecewise constant functions with respect to the mesh $\mathcal{M}$, *i.e.*,

$$\mathcal{E}_\mathcal{M} = \{\varphi : \Omega \to \mathbb{R}, \varphi(x) = \sum_{K\in\mathcal{M}} \varphi_K 1_K(x), \ \varphi_K \in \mathbb{R}\}.$$

For $\varphi \in \mathcal{E}_\mathcal{M}$, one defines the norm $\|.\|_\mathcal{M}$, by:

$$\|\varphi\|_\mathcal{M} = \sum_{K\in\mathcal{M}} \alpha_K |\varphi_K|,$$

where α_K is the time of flight in the control volume K, $\alpha_K = \sum_{l\in I_K} \Delta_K^l$.
This norm depends on the mesh $\mathcal{M}$ and also on the set $\mathcal{L}$. It is equivalent to the usual L^1 norm on $\mathcal{E}_\mathcal{M}$, more precisely, one has

Lemma 3.2 *Let $\varphi \in \mathcal{E}_{\mathcal{M}}$ and suppose that the assumption (*HML*) is satisfied. Then the following holds:*

$$\int_\Omega |\varphi| dx \leq \frac{2^d m(B(0,1)) h^{d-1}}{c_{inf}} \|\varphi\|_{\mathcal{M}}, \tag{15}$$

where $B(0,1)$ is the unit ball of $\mathbb{R}^d$.

Proof: On one hand, given the assumption on mesh (HML2) and relation (9), one has

$$\forall K \in \mathcal{M},\ \alpha_K \geq c_{inf} h \geq \frac{c_{inf} m(K)}{2^d m(B(0,1)) h^{d-1}},$$

yielding

$$\forall K \in \mathcal{M},\ m(K) \leq \frac{2^d m(B(0,1)) h^{d-1}}{c_{inf}} \alpha_K.$$

On the other hand:

$$\begin{aligned} \int_\Omega |\varphi| dx &= \sum_{K \in \mathcal{M}} \int_K |\varphi| dx = \sum_{K \in \mathcal{M}} m(K) |\varphi_K| \\ &\leq \frac{2^d m(B(0,1)) h^{d-1}}{c_{inf}} \sum_{K \in \mathcal{M}} \alpha_K |\varphi_K| = \frac{2^d m(B(0,1)) h^{d-1}}{c_{inf}} \|\varphi\|_{\mathcal{M}}, \end{aligned}$$

which completes the proof. ■

3.2 Maps between $\mathcal{E}_{\mathcal{M}}$ and $\mathcal{E}_{\mathcal{L}}$

In this part we give maps between $\mathcal{E}_{\mathcal{M}}$ and $\mathcal{E}_{\mathcal{L}}$. With these maps, we will be able to construct a function in $\mathcal{E}_{\mathcal{L}}$ from a given function in $\mathcal{E}_{\mathcal{M}}$ and vice versa. This step is very important in this method, in particular when the velocity depends on time. One should however note that we loose the conservativity with this method. For more details see remark 6 and [1].

Definition 3.3 *For any function $\psi \in \mathcal{E}_{\mathcal{M}}$ we associate a function $L(\psi) \in \mathcal{E}_{\mathcal{L}}$ to ψ defined by*

$$\forall l \in \mathcal{L},\ L(\psi)^l_j = \psi_K,\ \text{if } j \in I_{K,l} \tag{16}$$

and, conversely, if $\Phi = (\phi^l)_l \in \mathcal{E}_{\mathcal{L}}$, we define a function $M(\Phi) \in \mathcal{E}_{\mathcal{M}}$ by

$$\forall K \in \mathcal{M},\ M(\Phi)_K = \sum_{l \in I_K} \omega_l \left(\sum_{j \in I_{K,l}} (\Delta^l_j / \Delta^l_K) \psi^l_j \right), \tag{17}$$

where

$$\omega_l = \frac{\Delta^l_K}{\alpha_K}. \tag{18}$$

Proposition 3.4 *The maps M and L are linear, continous and 1-Lipschitz. Moreover we have:*

$$\forall \Phi \in \mathcal{E}_{\mathcal{M}}\ ,\quad M \circ L(\Phi) = \Phi\ , \tag{19}$$

$$\forall \Phi \in \mathcal{E}_{\mathcal{M}}\ ,\ \|\Phi\|_{\mathcal{M}} = \|L(\Phi)\|_{\mathcal{L}}\ , \tag{20}$$

$$\forall \Psi \in \mathcal{E}_{\mathcal{L}}\ ,\ \|M(\Psi)\|_{\mathcal{M}} \leq \|\Psi\|_{\mathcal{L}}. \tag{21}$$

Remark 6 *For the physical problem we are interested in, we need to have one constant value for each control volume for u (u is the saturation) to be able to compute velocity at the next step, i.e., to solve the pressure equation (see [12], [17]). For this reason we make this passage between $\mathcal{E}_{\mathcal{L}}$ and $\mathcal{E}_{\mathcal{M}}$, and thus we cannot be satisfied with a numerical solution defined on the streamlines (see [1]).*

3.3 Projections of $\mathcal{C}^1(\overline{\Omega},\mathbb{R})$ into $\mathcal{E}_\mathcal{M}$ and $\mathcal{E}_\mathcal{L}$

To be able to obtain error estimates between the numerical solution obtained by the streamline method and the exact solution of the problem, we introduce in this part the definitions of projections of a function belonging to $\mathcal{C}^1(\overline{\Omega},\mathbb{R})$ into the spaces $\mathcal{E}_\mathcal{L}$ and $\mathcal{E}_\mathcal{M}$, and we give some lemmas which will be used to show convergence (*c.f.* part 5.1).

Definition 3.5 *For all function $\theta \in \mathcal{C}^1(\overline{\Omega},\mathbb{R})$ we define*

- $\mathbb{P}_\mathcal{M}(\theta) \in \mathcal{E}_\mathcal{M}$ *given by*

$$\forall K \in \mathcal{M},\ (\mathbb{P}_\mathcal{M}(\theta))_K = \frac{1}{m_K}\int_K \theta(x)dx, \tag{22}$$

- $\mathbb{P}_\mathcal{L}(\theta) = (\theta^l)_{l\in\mathcal{L}} \in \mathcal{E}_\mathcal{L}$ *given by*

$$\forall l \in \mathcal{L}, \forall 0 \le j \le M_l - 1 :\ \theta^l_j = \frac{1}{\Delta^l_j}\int_{[\tau^l_j,\tau^l_{j+1})} \theta(q^l(\tau))d\tau. \tag{23}$$

Lemma 3.6 *Let $\theta \in \mathcal{C}^1(\overline{\Omega},\mathbb{R})$, one defines $\mathbb{P}_\mathcal{M}(\theta)$ and $\mathbb{P}_\mathcal{L}(\theta)$ as (22-23). Then there exists $c_\theta > 0$ depending only on $\|\overrightarrow{\nabla}\theta\|_{L^\infty(\Omega)}$, $\overrightarrow{V}$ and Ω, such that*

$$\|\mathbb{P}_\mathcal{L}(\theta) - L(\mathbb{P}_\mathcal{M}(\theta))\|_\mathcal{L} \le c_\theta N_\mathcal{L} h.$$

Proof: Let $l \in \mathcal{L}$. Let $u = (\mathbb{P}_\mathcal{L}(\theta))_l$ and $v = (L(\mathbb{P}_\mathcal{M}(\theta)))_l$. Thanks to the definition of the norm on l we have

$$\|u - v\|_l = \sum_{0\le j\le M_l-1} \Delta^l_j |u_j - v_j|.$$

Let $j \in \{0, ..., M_l - 1\}$. There exists only one control volume $K \in \mathcal{M}$, such that $\{q^l(\tau), \tau \in [\tau^l_j, \tau^l_{j+1})\} \subset K$. Then we have

$$u_j = \frac{1}{m(K)}\int_K \theta(x)dx, \quad v_j = \frac{1}{\Delta^l_j}\int_{\tau_j}^{\tau_{j+1}} \theta(q^l(\tau))d\tau,$$

which yields $|u_j - v_j| \le \|\overrightarrow{\nabla}\theta\|_{L^\infty(\Omega)} h$. Since θ belongs to $\mathcal{C}^1$, this gives us :

$$\|u - v\|_l \le \sum_{0\le j\le M_l-1} \Delta^l_j \|\overrightarrow{\nabla}\theta\|_{L^\infty(\Omega)} h \le |I_l| \|\overrightarrow{\nabla}\theta\|_{L^\infty(\Omega)} h.$$

Using relation (10), we see that $|I_l| < C_{\Omega,V}$ for all $l \in \mathcal{L}$, where $C_{\Omega,V}$ is a constant in $\mathbb{R}^*_+$ which depends on Ω and the velocity $\overrightarrow{V}$.
Now, let $c_\theta = C_{\Omega,V}\|\overrightarrow{\nabla}\theta\|_{L^\infty(\Omega)}$. According to above we deduce:

$$\|\mathbb{P}_\mathcal{L}(\theta) - L(\mathbb{P}_\mathcal{M}(\theta))\|_\mathcal{L} = \sum_{l\in\mathcal{L}} \|(\mathbb{P}_\mathcal{L}(\theta))_l - (L(\mathbb{P}_\mathcal{M}(\theta)))_l\|_l \le c_\theta N_\mathcal{L} h,$$

which gives the result. ■

Lemma 3.7 *Let $\theta \in \mathcal{C}^1(\overline{\Omega},\mathbb{R})$. We define $\mathbb{P}_\mathcal{M}(\theta)$ and $\mathbb{P}_\mathcal{L}(\theta)$ as in (22-23). For all $v = (v^l)_l \in \mathcal{E}_\mathcal{L}$ we have*

$$\|\mathbb{P}_\mathcal{M}(\theta) - M(v)\|_\mathcal{M} \le \|L(\mathbb{P}_\mathcal{M}(\theta)) - \mathbb{P}_\mathcal{L}(\theta)\|_\mathcal{L} + \|\mathbb{P}_\mathcal{L}(\theta) - v\|_\mathcal{L}$$

Proof: By proposition (3.4) we have

$$M \circ L(\mathbb{P}_{\mathcal{M}}(\theta)) = \mathbb{P}_{\mathcal{M}}(\theta).$$

Using relation (21) we deduce

$$\begin{aligned} \|\mathbb{P}_{\mathcal{M}}(\theta) - M(v)\|_{\mathcal{M}} &= \|M\left(L(\mathbb{P}_{\mathcal{M}}(\theta)) - v\right)\|_{\mathcal{M}} \\ &\leq \|L(\mathbb{P}_{\mathcal{M}}(\theta)) - v\|_{\mathcal{L}}. \end{aligned}$$

We add and substract $\mathbb{P}_{\mathcal{L}}(\theta)$ to obtain the desired result. ■

In the following, we write by $D = \mathbb{P}_{\mathcal{M}}(\mathrm{div}(\overrightarrow{V}))$ and $D_\infty = \|\mathrm{div}(\overrightarrow{V})\|_{L^\infty(\Omega)}$.

4 A streamline algorithm

We first give the numerical scheme along a streamline, then we present the algorithm of the streamline method.

4.1 Numerical solution on a streamline

Let $l \in \mathcal{L}$ be a streamline ($l = (p^l, I_l, q^l)$), and u the solution of problem $(\mathcal{P})$. The equation satisfied by u reduces on l to the following problem (*c.f.* 1.3):

$$\partial_t u_l(\tau,t) + \partial_\tau f(u_l(\tau,t)) + f(u_l(\tau,t))\mathrm{div}(\overrightarrow{V}(q_l(\tau)) = 0,\ \forall(\tau,t) \in I_l \times (0,T). \tag{24}$$

Equation (24) is completed by the following relations:

$$u_l(\tau,0) = u_0(q^l(\tau)),$$

$$u_l(0,t) = g(p^l,t),$$

where u_l is defined in (6). To simplify the notations we will note thereafter $v = u_l$.

Our goal is to compute an approximate solution of equation (24) on $I_l \times [0,T)$. To that propose a discretization $(\sigma^i)_{0 \leq i \leq N^l}$ of $[0,T)$ is given by:

$$\forall i \in \{0,..,N^l\}\ ,\ \sigma^i = i \times k^l \text{ with } k^l = \frac{T}{N^l}.$$

The initial data (at $\sigma^0 = 0$) on l is given by $v_0 = (L(\mathbb{P}_{\mathcal{M}}(u_0)))_l$,*i.e.*,

$$v_0(\tau) = u_{0,K} = \frac{1}{m(K)} \int_K u_0(x)dx, \quad \text{if } q^l(\tau) \in K.$$

The numerical solution is obtained using the following numerical scheme:

$$\begin{cases} \forall(j,s) \in \{0,..,M_l - 1\} \times \{0,..,N^l - 1\}, & v_j^{s+1} - v_j^s = \dfrac{k^l}{\Delta_j^l}(f_{j-1}^s - f_j^s) - k^l f_j^s D_j, \\ \forall j \in \{0,..,M_l - 1\}, & v_j^0 = \left(L(\mathbb{P}_{\mathcal{M}}(u_0))\right)_j^l, \\ \forall s \in \{0,..,N^l - 1\}, & v_{-1}^s = g(p^l,\sigma^s), \end{cases} \tag{25}$$

where g is a given data of problem $(\mathcal{P})$ (*c.f.* also assumptions (D)), v_j^s is expected to be an approximation of v on $[\tau_j^l, \tau_{j+1}^l) \times [\sigma^s, \sigma^{s+1})$, $f_j^s = f(v_j^s)$ is the approximation of $f(v(\tau_j,\sigma^s))$. k^l is the time step on l, k^l satisfies the following condition (CFL):

$$\exists 0 < k^l_m < k^l_M = \frac{\alpha^l}{\|f'\|_{L^\infty}} \text{ such that } : k^l_m \Delta^l \leq k^l \leq k^l_M \Delta^l, \tag{26}$$

where α^l is defined by the relation (11).
Finally, $D_j = D_K = \frac{1}{m(K)} \int_K \mathrm{div}(\overrightarrow{V}(x))dx$ if $\{q(\tau), \tau \in [\tau_j, \tau_{j+1})\} \subset K$. $v^{app,l}$ will be defined by

$$v^{app,l}(\tau,\sigma) = v^s_j \ , \quad \text{if } (\tau,\sigma) \in [\tau_j, \tau_{j+1}) \times [\sigma^s, \sigma^{s+1}).$$

In order to obtain error estimates for the streamline method, one defines w^l on $I_l \times [0,T]$ as the solution of the same numerical scheme (25) but with the initial data equal to $(\mathbb{P}_{\mathcal{L}}(u(.,0)))_l$.

Lemma 4.1 *Let $v^{app,l}$, w^l be defined by scheme (25). We suppose that the condition (26) on the time step is satisfied. Then:*

$$\|w^l(T) - v^{app,l}(T)\|_l \leq e^{(\|f'\|_{L^\infty} D_\infty T)} \|w^l(0) - v^{app,l}(0)\|_l.$$

Proof: To simplify the writing, one poses $w = w^l$ and $v = v^{app,l}$. For all $(j,s) \in \{-1,..,M_l-1\} \times \{0,..N^l\}$, one defines

$$U^s_j = w^s_j - v^s_j,$$

with $w^s_{-1} = v^s_{-1} = g(p^l, \sigma^s)$. One also defines:

$$F^s_j = \begin{cases} \dfrac{f(w^s_j) - f(v^s_j)}{w^s_j - v^s_j} & \text{if} \quad w^s_j \neq v^s_j, \\ f'(w^s_j) & \text{else.} \end{cases}$$

Considering $f' \geq 0$, one has $F^s_j \geq 0$ and $F^s_j \leq \|f'\|_\infty$.
Since w and v verify the same scheme (25), one deduces:

$$\forall (j,s) \in \{1,..,M_l-1\} \times \{0,..N^l-1\}, \ \Delta^l_j U^{s+1}_j = (\Delta^l_j - k^l F^s_j) U^s_j + k^l F^s_{j-1} U^s_j - k^l D_j F^s_j U^s_j.$$

k^l is selected such that $k^l \|f'\|_{L^\infty} \leq \Delta^l_j$ for all j (CFL), then one has $(\Delta^l_j - k^l F^s_j) \geq 0$. We then obtain

$$\forall (j,s) \in \{0,..,M_l-1\} \times \{0,..N^l-1\}, \ \Delta^l_j |U^{s+1}_j| \leq (\Delta^l_j - k^l F^s_j)|U^s_j| + k^l F^s_{j-1} |U^s_j| + k^l \Delta^l_j D_\infty F^s_j |U^s_j|.$$

Summing with respect to j in the preceding relation one obtains

$$\begin{aligned} \sum_{j=0}^{M_l-1} \Delta^l_j |U^{s+1}_j| &\leq (1 + k^l \|f'\|_{L^\infty} D_\infty) \sum_{j=0}^{M_l-1} \Delta^l_j |U^s_j| - k^l \sum_{j=0}^{M_l-1} F^s_j |U^s_j| + k^l \sum_{j=0}^{M_l-1} F^s_{j-1} |U^s_{j-1}| \\ &= (1 + k^l \|f'\|_{L^\infty} D_\infty) \sum_{j=0}^{M_l-1} \Delta^l_j |U^s_j| - k^l \sum_{j=0}^{M_l-1} F^s_j |U^s_j| + k^l \sum_{j=-1}^{M_l-2} F^s_j |U^s_j| \\ &= (1 + k^l \|f'\|_{L^\infty} D_\infty) \sum_{j=0}^{M_l-1} \Delta^l_j |U^s_j| - k^l F^s_{M_l-1} |U^s_{M_l-1}| \\ &\leq (1 + k^l \|f'\|_{L^\infty} D_\infty) \sum_{j=0}^{M_l-1} \Delta^l_j |U^s_j|. \end{aligned}$$

By induction on s, we find:

$$\|U^{N^l}\|_l \leq (1 + k^l\|f'\|_{L^\infty} D_\infty)^{N^l} \|U^0\|_l,$$

Note that $(1+k^l\|f'\|_{L^\infty} D_\infty)^{N^l} = (1+k^l\|f'\|_{L^\infty} D_\infty)^{\frac{T}{k^l}} \leq e^{(\|f'\|_{L^\infty} D_\infty T)}$. The result follows. ■

The following gives an error estimate between v, the solution of problem (24), and w^l.

Lemma 4.2 *Let v be the solution of problem (24) and w^l defined by the scheme (25) with $w(.,0) = (\mathbb{P}_{\mathcal{L}}(u(.,0)))_l$. We suppose that $v \in \mathcal{C}^2(I_l \times \mathcal{I})$ and that assumptions (*`HML`*) and the condition (26) on the time step are satisfied. Then there exists a constant $c_{vf} \in \mathbb{R}_+^*$, which depends on f, u (and its derivative of order 1 and 2), and g such that:*

$$\|v(.,T) - w^l(T)\|_{L^1(I_l)} \leq c_{vf} h_l. \tag{27}$$

Proof: The proof of this lemma will be given in appendix A. ■

Remark 7 *In lemma 4.2 we use the assumption that u (u is the solution of the problem and $v = u_{|_l}$) is of class $\mathcal{C}^2$, to obtain an error estimate of order h_l.*

4.2 Algorithm

The algorithm to construct an approximate solution of the problem $(\mathcal{P})$ by the streamline method is as follows:

1. one traces streamlines,
2. one computes $u^{app}(0) = \mathbb{P}_{\mathcal{M}}(u_0) \in \mathcal{E}_{\mathcal{M}}$,
3. for all $l \in \mathcal{L}$, one computes $(v^{app,l}(T))$,
4. one determines $v^{app}(T) = (v^{app,l}(T))_{l\in\mathcal{L}} \in \mathcal{E}_{\mathcal{L}}$,
5. one calculates $u^{app}(T) = M(v^{app}(T)) \in \mathcal{E}_{\mathcal{M}}$, (see relation 17).

5 Main result

The objective of this part is to show some properties of the streamline method. One will denote by $(\mathcal{H})$ the following assumptions:

$(\mathcal{H}1)$ The data of problem $(\mathcal{P})$ satisfies assumption (`D`).

$(\mathcal{H}2)$ The mesh $\mathcal{M}$ on Ω is supposed to be regular.

$(\mathcal{H}3)$ Relation (10) is satisfied.

$(\mathcal{H}4)$ Assumptions (`HML`) between the mesh $\mathcal{M}$ and the set of the meshs over each streamline are satisfied.

$(\mathcal{H}5)$ Over each streamline, condition (CFL) on the time step is satisfied.

5.1 A convergence result

Let u be the exact solution of problem $(\mathcal{P})$, assumed to be of class $\mathcal{C}^2$ on Q. For $t \in [0,T)$ one defines $\mathbb{P}_{\mathcal{M}}(u)(t) = \mathbb{P}_{\mathcal{M}}(u(.,t)) \in \mathcal{E}_{\mathcal{M}}$ as in relation (22) and $\mathbb{P}_{\mathcal{L}}(u)(t) = \mathbb{P}_{\mathcal{L}}(u(.,t)) \in \mathcal{E}_{\mathcal{L}}$ as in (23).

Lemma 5.1 *Let u be the solution of problem $(\mathcal{P})$ and u^{app} the approximate solution obtained by the streamline method. We suppose that $u \in \mathcal{C}^2(Q)$ and that assumptions $(\mathcal{H})$ are satisfied. Then one has:*

$$\|(\mathbb{P}_{\mathcal{M}}(u) - u^{app})(T)\|_{\mathcal{M}} \leq C_* \times N_{\mathcal{L}} h,$$

where $C_* \in \mathbb{R}_+^*$ depends on $\|\overrightarrow{\nabla}_x u\|_{L^\infty(\Omega\times(0,T))}$, g, f, u_0, T , $\overrightarrow{V}$ (also $div(\overrightarrow{V})$) and Ω.

Remark 8 *Lemma 5.1, gives us an estimate of the norm $\|.\|_{\mathcal{M}}$ but not a result of convergence. In particular, if $d = 2$ (see assumption* `HML3`*) one has $N_{\mathcal{L}} \leq \frac{c}{h}$ with $c \in \mathbb{R}_+^*$, which only gives*

$$\|(\mathbb{P}_{\mathcal{M}}(u) - u^{app})(T)\|_{\mathcal{M}} \leq C.$$

Proof: Define

$$\begin{aligned} A(t) &= \|\mathbb{P}_{\mathcal{M}}(u)(t) - u^{app}(t)\|_{\mathcal{M}} \ , \\ B(t) &= \|\mathbb{P}_{\mathcal{L}}(u)(t) - L(\mathbb{P}_{\mathcal{M}}(u))(t)\|_{\mathcal{L}}, \\ C(t) &= \|\mathbb{P}_{\mathcal{L}}(u)(t) - v^{app}(t)\|_{\mathcal{L}}. \end{aligned}$$

By lemma (3.7), and using $u^{app}(T) = M(v^{app})(T)$ one finds

$$\begin{aligned} A(T) &= \|\mathbb{P}_{\mathcal{M}}(u)(T) - M(v^{app})(T)\|_{\mathcal{M}} \\ &\leq \|\mathbb{P}_{\mathcal{L}}(u)(T) - L(\mathbb{P}_{\mathcal{M}}(u))(T)\|_{\mathcal{L}} + \|\mathbb{P}_{\mathcal{L}}(u)(T) - v^{app}(T)\|_{\mathcal{L}} \\ &= B(T) + C(T). \end{aligned}$$

Introducing $W = (w^l)_{l\in\mathcal{L}}$ (defined in 4.1) one obtains

$$C(T) \leq \|(\mathbb{P}_{\mathcal{L}}(u) - W)(T)\|_{\mathcal{L}} + \|(W - v^{app})(T)\|_{\mathcal{L}}. \tag{28}$$

By lemma (4.1) one can write:

$$\|(W - v^{app})(T)\|_{\mathcal{L}} \leq e^{(\|f'\|_{L^\infty} D_\infty T)} \|(W - v^{app})(0)\|_{\mathcal{L}}.$$

According to the definition of W and v^{app} one has:

$$\begin{aligned} W(0) &= \mathbb{P}_{\mathcal{L}}(u)(0)\,; \\ v^{app}(0) &= L(\mathbb{P}_{\mathcal{M}}(u))(0). \end{aligned}$$

One therefore obtains

$$\|(W - v^{app})(T)\|_{\mathcal{L}} \leq e^{(\|f'\|_{L^\infty} D_\infty T)} \|(\mathbb{P}_{\mathcal{L}}(u) - \mathcal{L}(\mathbb{P}_{\mathcal{M}}(u)))(0)\|_{\mathcal{L}} = e^{(\|f'\|_{L^\infty} D_\infty T)} B(0).$$

Estimation of $B(t)$:

According to lemma (3.6) there exists a constant $c_u \in \mathbb{R}_+^*$, which depends on $\|\overrightarrow{\nabla}_x u\|_{L^\infty(\overline{\Omega}\times(0,T))}$ $\overrightarrow{V}$ and Ω , such that

$$\forall t \in [0,T] \ , \ B(t) \leq c_u N_{\mathcal{L}} h. \tag{29}$$

Estimation of $\|(\mathbb{P}_{\mathcal{L}}(u)-W)(T)\|_{\mathcal{L}}$:

By lemma (4.2) there exists a constant $c_2 \in \mathbb{R}_+^*$ which depends on $\|\overrightarrow{\nabla} u_0\|_{L^\infty(\Omega)}$, g ,$\overrightarrow{V}$, u and f, such that

$$\|(\mathbb{P}_{\mathcal{L}}(u)-W)(T)\|_{\mathcal{L}} \leq c_2 \sum_{l\in\mathcal{L}} h_l. \tag{30}$$

Assumption (HML2) gives

$$\forall l \in \mathcal{L},\ h_l \leq \beta\Delta^l \leq c_{max}\beta h \tag{31}$$

By replacing (31) in (30) one obtains

$$\|(\mathbb{P}_{\mathcal{L}}(u)-W)(T)\|_{\mathcal{L}} \leq c_2 c_{max}\beta N_{\mathcal{L}} h. \tag{32}$$

By using relations (29) and (32), one finally finds

$$\begin{aligned} A(T) &\leq B(T) + e^{(\|f'\|_{L^\infty} D_\infty T)} B(0) + c_2 c_{max}\beta N_{\mathcal{L}} h \\ &\leq \left(c_u + c_u e^{(\|f'\|_{L^\infty} D_\infty T)} + c_2 c_{max}\beta\right) N_{\mathcal{L}} h. \end{aligned}$$

Hence the result with $C_* = \left(c_u + c_u e^{(\|f'\|_{L^\infty} D_\infty T)} + c_2 c_{max}\beta\right)$. ■

Theorem 5.2 *Let u be the solution of problem $(\mathcal{P})$. Let u^{app} be the approximate solution of problem $(\mathcal{P})$ obtained by the streamline method. One assumes that $u \in \mathcal{C}^2(Q)$ and that assumptions $(\mathcal{H})$ are satisfied. Then, there exists a constant $c \in \mathbb{R}_+^*$ such that:*

$$\int_\Omega |u(x,T) - u^{app}(x,T)| dx \leq ch,$$

where c depends on u_0, g, f, $\overrightarrow{V}$, u and on the mesh defined on Ω (i.e. c_{max}, c_{NLDC},...)

Proof: Let $\mathbb{P}_{\mathcal{M}}(u)(T) \in \mathcal{E}_{\mathcal{M}}$, as defined in (22). Then

$$\int_\Omega |u(x,T) - u^{app}(x,T)| dx \leq \int_\Omega |u(x,T) - \mathbb{P}_{\mathcal{M}}(u)(x,T)| dx + \int_\Omega |\mathbb{P}_{\mathcal{M}}(u)(x,T) - u^{app}(x,T)| dx. \tag{33}$$

However, since $u \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R})$, there exists a constant $c_{\mathcal{M}} \in \mathbb{R}_+^*$ which depends on $\|\overrightarrow{\nabla}_x u(.,T)\|_{L^\infty(\Omega)}$, such that

$$\int_\Omega |u(x,T) - \mathbb{P}_{\mathcal{M}}(u)(x,T)| dx \leq c_{\mathcal{M}} h. \tag{34}$$

One notes that

$$\int_\Omega |\mathbb{P}_{\mathcal{M}}(u)(x,T) - u^{app}(x,T)| dx \leq \frac{h^{d-1}}{c_{inf}} \|\mathbb{P}_{\mathcal{M}}(u)(T) - u^{app}(T)\|_{\mathcal{M}}. \tag{35}$$

By using (HML3) and the result of lemma 5.1, relation (35) becomes (one assumes that $h \leq 1$)

$$\int_\Omega |\mathbb{P}_{\mathcal{M}}(u)(x,T) - u^{app}(x,T)| dx \leq C_* h \tag{36}$$

(36) and (34) in (33) give

$$\int_\Omega |u(x,T) - u^{app}(x,T)| dx \leq ch,$$

with $c = (c_{inf} \times c_{NLDC}(c_{\mathcal{M}} + c_{vf} c_{max}\beta)) + c_{\mathcal{M}}$. ■

6 Numerical tests

We present two numericals tests of the streamline method. For these tests, we present a comparison with the exact solution.

6.1 Test 1

Let $\Omega = (1,2) \subset \mathbb{R}$ and $I = [0,5]$. We use the following input data:

- the velocity is given by
$$\forall x \in \Omega,\ \overrightarrow{V}(x) = 1/x,$$
- the initial data at $t = 0$
$$\forall x \in \Omega,\ u_0(x) = x \times e^{-\frac{x}{2}},$$
- the boundary condition (at point $x = 1$) :
$$\forall t \in I,\ g(1,t) = \sqrt{e}\, e^t.$$

With these data, the solution of problem $(\mathcal{P})$ is:

$$\forall x \in \Omega,\ t \in I,\quad u(x,t) = xe^{-\frac{x}{2}} e^t.$$

In this case, it is sufficient to trace only one streamline. After computation, one finds:

$$\forall \tau \in [0, 3/2],\ q(\tau) = \sqrt{2\tau + 1} \text{ and } q(0) = 1,$$

which gives

$$v(\tau,t) = u(q(\tau),t) = \sqrt{2\tau+1}\, e^{t-\tau-\frac{1}{2}}.$$

The table below shows the error in the L^1 norm ($Er = \|u(.,T) - u^{app}(T)\|_{L^1(\Omega)}$).

h	Er	$\frac{\log(Er(h)) - \log(Er(h_0))}{\log(h) - \log(h_0)}$
h_0=0.0025	0.97873	-
0.0010	0.38966	1.0051
0.0020	0.07774	1.0028
0.0010	0.03886	1.0023
0.0002	0.00777	1.0016
0.0001	0.00388	1.0014

6.2 Test 2

One considers $\Omega = (1,11) \times (0,10) \subset \mathbb{R}^2$, $I = [0,1)$ the time interval and the other data of the test are:

- the velocity is given by:
$$\forall (x,y) \in \overline{\Omega}\,,\ \overrightarrow{V}(x,y) = (x,y),$$
and $\overrightarrow{V}$ satisfies $\forall (x,y) \in \Omega$, $\mathrm{div}(\overrightarrow{V}(x,y)) = 2$.
- the initial data at $t = 0$:
$$\forall (x,y) \in \Omega\,,\ u_0(x,y) = xye^7,$$

- for the boundary condition at $\partial\Omega$, one assumes that:

$$g(x_r, y_r, t) = x_r y_r e^{-4t+7} \ , \forall (x_r, y_r, t) \in \Sigma_-.$$

With these data, the exact solution of problem $(\mathcal{P})$ is given by:

$$\forall (x, y, t) \in Q = \Omega \times I \ , u(x, y, t) = xye^{-4t+7}.$$

At $t = T$, the L^1 and L^2 norms of u in Ω are

$$\|u(., T)\|_{L^1(\Omega)} = 5.02138, \ \|u(., T)\|_{L^2(\Omega)} = 6.69518.$$

In this test one takes a Cartesian grid on Ω made up of squares.

The table below shows the error in the L^1 norm, and (Fig-3) shows the set of streamlines traced in Ω.

h	Er	$\frac{\log(Er(h))-\log(Er(h_0))}{\log(h)-\log(h_0)}$
h_0=0.04710	0.06379	-
0.03536	0.04307	1.3661
0.01768	0.01797	1.2917
0.00884	0.00793	1.2461
0.00707	0.00612	1.2359
0.00442	0.00365	1.2092

One notes that the method behaves well and that one has a very good precision of the approximate solution with respect to the exact solution.

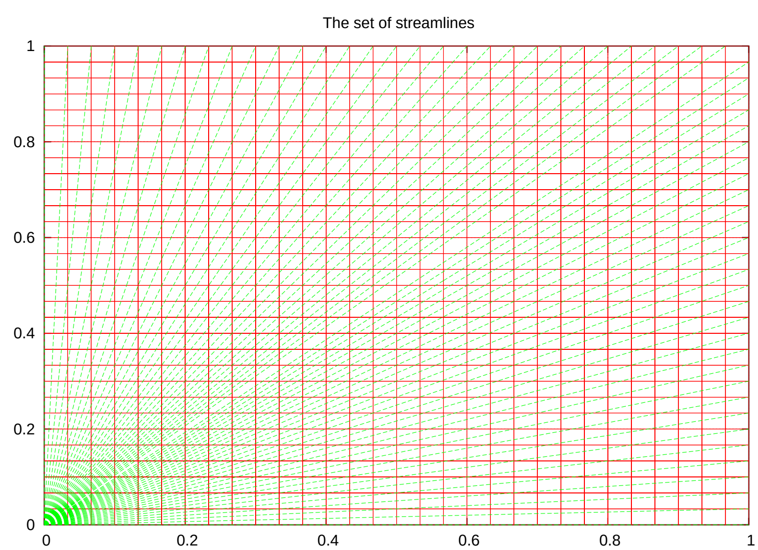

Fig 3. *The mesh $\mathcal{M}$ with the set of streamlines traced in Ω, Test 2*

7 Conclusion and perspective:

To conclude, in the present paper the convergence of the streamline method has been proved for $\mathcal{C}^2$ solutions of problem $(\mathcal{P})$, and numerical tests have been provided which confirm such theoretical results.

In perspective, it would be interesting to extend such convergence result to the case of a function u which belongs to BV.
The final goal of this research line would be to analyze the case where the velocity also depends on time.

A Proof of lemma 4.2

To simplify the notation we remove the index l, keeping in mind the fact that the parameters given hereafter depend on the streamline l.

Recall that l is given by $l = (p, q, I)$ such that $q(0) = p$ and $\overrightarrow{\frac{dq}{d\tau}}(\tau) = \overrightarrow{V}(q(\tau))$, a subdivision $\Gamma = (\tau_i)_{0\le i\le M}$ of I is given and Γ satisfies the condition (11).
The goal of this part is to give an error estimate between the approximate solution w defined by scheme (25) and the exact solution v ($v \in \mathcal{C}^2(I \times \mathcal{I})$).
From the exact solution v of equation (24) we define v_h by

$$v_h(\tau, t) = v_i^n, \ \text{ if } (\tau, t) \in]\tau_i, \tau_D ii + 1] \times [t^n, t^{n+1}), \tag{37}$$

with $(v_i^n)_{i,n}$ defined by

$$\forall\, 0 \le i < M,\ 0 \le n \le N,\ \ v_i^n = v(\tau_{i+1}, t^n). \tag{38}$$

To complete the definition of v_h, one adds the following data:

$$\forall n \in \{0, ..., N\}\,,\ v_{-1}^n = v_h(0, t^n) = v(0, t^n) = g(p, t^n).$$

Lemma A.1 *Let $(v_i^n)_{i,n}$ defined by relation (38). Then, for all $i \in \{0, ..., M\}$ and $n \in \{0, ..., N-1\}$ one has*

$$v_i^{n+1} = v_i^n - \frac{k}{\Delta_i}\left(f(v_i^n) - f(v_{i-1}^n)\right) - kD_i f(v_i^n) + C_i^n.$$

Furthermore, under assumption (`HML2`*) of part 2.2 and relation (26) on k, there exists a constant $C^0 \in \mathbb{R}_+$ which depends on f, g, u_0, v, c_{inf} and $\overrightarrow{V}$ such that*
$\forall (i, n) \in \{0, ..., M-1\} \times \{0, ..., N-1\}$, $|C_i^n| \le C^0\Delta^2$.

Proof: For all $i \in \{0, ..., M-1\}$, $n \in \{0, ..., N-1\}$ one defines

$$\mathcal{A}_i^n = \int_{t^n}^{t^{n+1}} \int_{\tau_i}^{\tau_{i+1}} \partial_t v(\tau, t) d\tau dt. \tag{39}$$

Since v satifies equation (24) one has:

$$\begin{aligned}\mathcal{A}_i^n &= -\int_{t^n}^{t^{n+1}} \int_{\tau_i}^{\tau_{i+1}} \left(\partial_\tau (f \circ v)(\tau, t) + f(v(\tau, t))\operatorname{div}(\overrightarrow{V}(q(\tau)))\right) d\tau dt.\\ &= -\int_{t^n}^{t^{n+1}} [f(v(\tau_{i+1}, t)) - f(v(\tau_i, t))]\, dt - \int_{t^n}^{t^{n+1}} \int_{\tau_i}^{\tau_{i+1}} f(v(\tau, t))\operatorname{div}(\overrightarrow{V}(q(\tau)))d\tau dt.\end{aligned}$$

However, the function $t \in \mathcal{I} \longrightarrow f \circ v(., t)$ is of classe $\mathcal{C}^2$ on $\mathcal{I}$, by Taylor's formula, one then has (for $\alpha \in \{i, i+1\}$ and $i > 0$ and $t \in \mathcal{I}$)

$$f(v(\tau_\alpha, t)) = f(v(\tau_\alpha, t^n)) + (t - t^n)\partial_t (f \circ v)(\tau_\alpha, t^n) + \int_{t^n}^{t} (t - \sigma)\partial_{tt}^2 (f \circ v)(\tau_\alpha, \sigma) d\sigma. \tag{40}$$

For the case $i = 0$ one has, for all $t \in \mathcal{I}$

$$f(v(\tau_0, t)) = f(g(q(0), t^n)) + (t - t^n)\partial_t (f \circ g)(q(0), t^n) + \int_{t^n}^{t} (t - \sigma)\partial_{tt}^2 (f \circ g)(q(0), \sigma) d\sigma. \tag{41}$$

By integrating the relation (40) between t^n and t^{n+1} one finds

$$\begin{aligned}\mathcal{A}_i^n &= -k(f(v(\tau_{i+1},t^n)) - f(v(\tau_i,t^n))) - k\Delta_i D_i f(v_i^n) + RT_i^n + RD_i^n \\ &= -k(f(v_i^n) - f(v_{i-1}^n)) - k\Delta_i D_i f(v_i^n) + RT_i^n + RD_i^n,\end{aligned} \tag{42}$$

where:

$$\begin{aligned}RT_i^n &= \frac{-k^2}{2}(\partial_t(f\circ v)(\tau_{i+1},t^n) - \partial_t(f\circ v)(\tau_i,t^n)) \\ &-\int_{t^n}^{t^{n+1}}\left(\int_{t^n}^{t}(t-\sigma)[\partial_{tt}^2(f\circ v)(\tau_{i+1},\sigma) - \partial_{tt}^2(f\circ v)(\tau_i,\sigma)]d\sigma\right)dt.\end{aligned} \tag{43}$$

$$\begin{aligned}RD_i^n &= -\int_{t^n}^{t^{n+1}}\int_{\tau_i}^{\tau_{i+1}} f(v(\tau,t))\mathrm{div}(\overrightarrow{V}(q(\tau)))d\tau dt + k\Delta_i D_i f(v_i^n) \\ &= -\int_{t^n}^{t^{n+1}}\int_{\tau_i}^{\tau_{i+1}}\Big((f(v(\tau,t)) - f(v_i^n))D_i + f(v(\tau,t)(\mathrm{div}(\overrightarrow{V}) - D_i)\Big)d\tau dt\end{aligned} \tag{44}$$

One notes that

$$\begin{aligned}\mathcal{A}_i^n &= \int_{\tau_i}^{\tau_{i+1}}\int_{t^n}^{t^{n+1}}\partial_t v(\tau,t)dtd\tau. \\ &= \int_{\tau_i}^{\tau_{i+1}}[v(\tau,t^{n+1}) - v(\tau,t^n)]d\tau.\end{aligned}$$

Taylor's formula applied to the function $\tau \to v(\tau,t^\alpha)$, yields

$$\forall\tau\in[\tau_i,\tau_{i+1}],\ v(\tau,t^\alpha) = v(\tau_{i+1},t^\alpha) + (\tau-\tau_{i+1})\partial_\tau v(\tau_{i+1},t^\alpha) + \int_{\tau_{i+1}}^{\tau}(\tau-\sigma)\partial_{\tau\tau}^2 v(\sigma,t^\alpha)d\sigma,$$

where $\alpha\in\{n,n+1\}$.
By integrating the relation above between τ_i and τ_{i+1} for $\alpha = n,\ n+1$, and substracting the two expressions, one obtains

$$\begin{aligned}\mathcal{A}_i^n &= \Delta_i(v(\tau_{i+1},t^{n+1}) - v(\tau_{i+1},t^n)) \\ &= \Delta_i(v_i^{n+1} - v_i^n) + RH_i^n,\end{aligned} \tag{45}$$

where:

$$\begin{aligned}RH_i^n &= \frac{-\Delta_i^2}{2}(\partial_\tau v(\tau_{i+1},t^{n+1}) - \partial_\tau v(\tau_{i+1},t^n)) \\ &+\int_{\tau_i}^{\tau_{i+1}}\int_{\tau_{i+1}}^{\tau}(\tau-\sigma)[\partial_{\tau\tau}^2 v(\sigma,t^{n+1}) - \partial_{\tau\tau}^2 v(\sigma,t^n)]d\sigma dx.\end{aligned} \tag{46}$$

Making relations (42) and (45) equal one finds

$$\begin{aligned}\forall 0\le i<M, 0\le n<N,\quad v_i^{n+1} = &\ v_i^n - \frac{k}{\Delta_i}(f(v_i^n) - f(v_{i-1}^n)) - kD_i f(v_i^n) \\ &+\frac{1}{\Delta_i}(RT_i^n - RH_i^n + RD_i^n),\end{aligned} \tag{47}$$

where $v_{-1}^n = v(0,t^n) = g(q(0),t^n)$ for all $n\in\{0,...,N\}$.

Estimate of $RT_i^n + RH_i^n$

Define $C_1 = \|\partial^2_{\tau t}(f\circ v)\|_{L^\infty}$, $C_2 = \|\partial^2_{t\tau} v\|_{L^\infty}$, $C_3 = \|\partial^2_{tt}(f\circ v)\|_{L^\infty}$, and $C_4 = \|\partial^2_{\tau\tau} f\circ v\|_{L^\infty}$. Since f and v are in $\mathcal{C}^2$, one has $C_1<\infty$, $C_2<\infty$, $C_3<\infty$ and $C_4<\infty$.
Let $(i,n)\in\{0,...,M-1\}\times\{0,...,N-1\}$. One starts by giving an estimate of RT_i^n. By the mean value theorem there exists $\alpha_i\in(\tau_i,\tau_{i+1})$ such that:

$$\partial_t(f\circ v)(\tau_{i+1},t^n) - \partial_t(f\circ v)(\tau_i,t^n) = \Delta_i\partial_\tau\partial_t(f\circ v)(\alpha_i,t^n).$$

This yields

$$|\partial_t(f\circ v)\tau_{i+1},t^n) - \partial_t(f\circ v)(\tau_i,t^n)| \le \Delta_i C_1.$$

Note that

$$\begin{aligned}|\int_{t^n}^{t^{n+1}}\int_{t^n}^{t}(t-\sigma)[\partial^2_{tt}(f\circ v)(\tau_{i+1},\sigma) - \partial^2_{tt}(f\circ v)(\tau_i,\sigma)]d\sigma dt| &\le \\ \int_{t^n}^{t^{n+1}}\int_{t^n}^{t}(t-\sigma)|\partial^2_{tt}(f\circ v)(\tau_{i+1},\sigma) - \partial^2_{tt}(f\circ v)(\tau_i,\sigma)|d\sigma dt &\le \frac{C_3k^3}{3},\end{aligned}$$

which finally gives

$$|RT_i^n| \le \Delta_i k^2\frac{C_1}{2} + k^3\frac{C_3}{3}. \tag{48}$$

A similar computation yields

$$|RH_i^n| \le \Delta_i^2 k\frac{C_2}{2} + \Delta_i^3\frac{C_4}{3}. \tag{49}$$

According to the assumptions made on the grid and the time step k one has

$$\forall 0\le i<M,\ k\le\Delta_i\le\Delta,$$

$$\begin{aligned}|RT_i^n| + |RH_i^n| &\le \Delta_i\left(\frac{C_1}{2\|f'\|^2_{L^\infty}} + \frac{C_3}{3\|f'\|^3_{L^\infty}}\right)\Delta^2 \\ &\quad +\Delta_i\left(\frac{C_2}{2\|f'\|_{L^\infty}} + \frac{C_4}{3}\right)\Delta^2.\end{aligned}$$

Defining $C^{0,1} = \left(\frac{C_1}{2\|f'\|^2_{L^\infty}} + \frac{C_3}{3\|f'\|^3_{L^\infty}} + \frac{C_2}{2\|f'\|_{L^\infty}} + \frac{C_4}{3}\right)$ one finds the following result:

$$\forall(i,n)\in\{0,...,M-1\}\times\{0,...N-1\},\ \ \frac{1}{\Delta_i}(|RT_i^n| + |RH_i^n|) \le C^{0,1}\Delta^2. \tag{50}$$

Estimate of RD_i^n

Since $f\circ v\in\mathcal{C}^2(I\times\mathcal{I},\mathbb{R})$ one then has:

$$\forall(\tau,t)\in[\tau_i,\tau_{i+1}]\times[t^n,t^{n+1}],\ |f(v(\tau,t)) - f(v(\tau_{i+1},t^n))| \le \|\vec{\nabla} f\circ v\|_{L^\infty}\sqrt{\Delta_i^2+k^2}.$$

However $k\le\frac{\Delta_i}{\|f'\|_{L^\infty}}$ (*c.f.* part 4.1, relation 26). This gives

$$\forall(\tau,t)\in[\tau_i,\tau_{i+1}]\times[t^n,t^{n+1}],\ |f(v(\tau,t)) - f(v(\tau_{i+1},t^n))| \le \sqrt{1+(\frac{1}{\|f'\|_{L^\infty}})^2}\|\vec{\nabla} f\circ v\|_{L^\infty}\Delta_i.$$

One notes that $\mathrm{div}(\overrightarrow{V}) \in \mathcal{C}^1(\Omega, \mathbb{R})$ (since $\overrightarrow{V} \in \mathcal{C}^2(\overline{\Omega}, \mathbb{R}^d)$). Then one obtains

$$\forall K \in \mathcal{M}, \forall x \in K, |\mathrm{div}(\overrightarrow{V}(x)) - D_K| \leq \|\overrightarrow{\nabla}(\mathrm{div}\overrightarrow{V})\|_{L^\infty} h,$$

where $D_K = \frac{1}{m(K)} \int_K \mathrm{div}(\overrightarrow{V}(x))dx$ and h is the mesh step on $\mathcal{M}$.
Using assumptions (HML2) (*c.f.* part 2.2) one obtains

$$\forall \tau \in [\tau_i, \tau_{i+1}],\ |\mathrm{div}(\overrightarrow{V}(q(\tau))) - D_i| \leq \|\overrightarrow{\nabla}(\mathrm{div}(\overrightarrow{V}))\|_{L^\infty} h \leq \|\overrightarrow{\nabla}(\mathrm{div}(\overrightarrow{V}))\|_{L^\infty} \frac{\Delta_i}{c_{inf}}.$$

According to what preceedes, one deduces

$$\forall (i, n) \in \{0, ..., M-1\} \times \{0, ...N-1\},\ \frac{1}{\Delta_i} |RD_i^n| \leq C^{0,2} \Delta^2, \tag{51}$$

where $C^{0,2}$ depends on $\overrightarrow{\nabla}\mathrm{div}$, $\overrightarrow{\nabla} f \circ v$, $\|f \circ v\|_{L^\infty}$ and c_{inf}.
Define $C^0 = C^{0,1} + C^{0,2}$. Then one obtains

$$\forall 0 \leq i < M,\ 0 \leq n < N\ : \frac{1}{\Delta_i}(|RT_i^n| + |RH_i^n| + |RD_i^n|) \leq C^0 \Delta^2,$$

which concludes the proof of the lemma. ■

Lemma A.2 *Under assumptions (*HLM2*) and (26) on the time step. There exists a constant* $C^1 \in \mathbb{R}_+^*$ *such that*

$$\|w - v_h(., T)\|_{L^1(I)} \leq C^1 \Delta,$$

where C^1 *depends on* v_0, g, f, $\overrightarrow{V}$ *and on the mesh on* I *(in particular on* k_m *defined in relation (26)).*

Proof: For all $n \in \{0, ..., N\}$ one defines

$$E^n = \sum_{0 \leq i < M} \Delta_i |w_i^n - v_i^n| = \int_I |(w - v_h)(\tau, t^n)| d\tau.$$

By following the same method used in the proof of lemma 4.1 one finds:

$$E^n \leq (1 + k\|f^{'}\|_{L^\infty} D_\infty) E^{n-1} + C^0 |I| \Delta^2,$$

where C^0 is the same constant found in lemma A.1.
By induction on n one finds

$$E^N \leq (1 + k\|f^{'}\|_{L^\infty} D_\infty)^N E^0 + C^0 |I| \left(\sum_{0 \leq j < N} (1 + k\|f^{'}\|_{L^\infty} D_\infty)^j \right) \Delta^2 \tag{52}$$

In addition one has

$$k = \frac{T}{N},\ (1 + k\|f^{'}\|_{L^\infty} D_\infty)^j \leq e^{(\|f^{'}\|_{L^\infty} D_\infty T)}.$$

Using relation (26) one deduces

$$E^N \leq e^{(\|f^{'}\|_{L^\infty} D_\infty T)} \left(E^0 + C|I| k_m \Delta \right). \tag{53}$$

Estimate of E^0:
At $t=0$, $w_j^0 = \frac{1}{\Delta_j}\int_{\tau_j}^{\tau_{j+1}} v_0(\tau)d\tau$ et $v_j^0 = v_0(\tau_{j+1})$. Since v_0 is of class $\mathcal{C}^2$ on I one deduces

$$\forall 0 \leq j < M,\ |w_j^0 - v_j^0| \leq C_0\Delta_j,$$

where C_0 is a positive constant that depends on $v_0^{'}$. By summing over j one obtains

$$E^0 \leq C_0|I|\Delta. \tag{54}$$

Now using relation (54) in (53) one finds

$$E^N \leq C^1\Delta,$$

where $C^1 = e^{(\|f^{'}\|_{L^\infty} D_\infty T)}\left(C_0 + C^0|I|k_m\right)$. The result follows. ■

Proof of lemma 4.2:
Let v be the exact solution of equation (24), w the solution of numerical scheme. One defines v_h by relations (37) and (38). One has:

$$\int_I |v(\tau,T) - w(\tau,T)|d\tau \leq \int_I |v(\tau,T) - v_h(\tau,T)|d\tau + \int_I |v_h(\tau,T) - w(\tau,T)|d\tau.$$

Since v is $\mathcal{C}^2$, there exists a constant $C^2 \in \mathbb{R}_+^*$ depending on $\|\partial_\tau v(.,T)\|_{L^\infty}$ such that

$$\int_I |v(\tau,T) - v_h(\tau,T)|d\tau \leq C^2\Delta.$$

By using the relation above and the result of lemma A.2 one finds

$$\int_I |v(\tau,T) - w(\tau,T)|d\tau \leq (C^1 + C^2)\Delta,$$

which gives the result.

Acknowledgment. The author would like to thank Mr T.Gallouët and Mr J. Wendebourg for their attention to this work and their advices and useful discussions

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