Finite volume schemes for a non linear hyperbolic conservation law with a flux function involving discontinuous coefficients

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Abstract

A model for two phase flow in porous media with distinct permeabilities leads to a non linear hyperbolic conservation law with a discontinuous flux function. In this paper for such a problem, the notion of entropy solution is presented and existence and convergence of a finite volume scheme are proved. No hypothesis of convexity or genuine non linearity on the flux function is assumed, which is a new point in comparison with preceeding works. As the trace of the solution along the line of discontinuity of the flux function can be considered, this problem is more complex. To illustrate these results, some numerical tests are presented.

ey words : discontinuous flux, entropy solution, finite volume schemes.

1 Introduction

The notion of entropy solution, and the convergence of finite volume scheme are presented for the following hyperbolic conservation law:

$$\begin{cases} \partial_t u + \partial_x\big(k(x)g(u) + f(u)\big) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x)\,, \end{cases} \tag{1}$$

with initial value $u_0 \in L^\infty(\mathbb{R};[0,1])$. And finally, several numerical results are introduced.
The functions f, g and k are supposed to satisfy the following hypotheses :

(H1) $g \in \mathrm{Lip}([0,1])$ is non-negative and $g(0) = g(1) = 0$,
(H2) $f \in \mathrm{Lip}([0,1])$,
(H3) k is the discontinuous function defined by

$$k(x) = \begin{cases} k_L & \text{if } x < 0 \\ k_R & \text{if } x > 0 \end{cases} \quad \text{with } k_L, k_R > 0 \text{ and } k_L \neq k_R\,.$$

The particular shape of the functions f, g and k described through the hypotheses (H1), (H2), (H3) is given by a model for two-phase flow in porous media with distinct permeabilities (see [Bac04]). Let us just claim here that, in this context, the hypotheses on f, g and k are natural.

No hypothesis of convexity or genuine non-linearity on g is assumed, which is a new point in comparison with all the preceding works on the subject (see by example [Tow00, Tow01, KT04, SV03, Bac04]). Indeed, these preceding works assume that the entropy solution must have traces along the line $\{x = 0\}$. To guarantee the existence of these traces, they impose that g is genuinely non linear. Without the hypothesis on g genuinely non linear, these traces of function can not be considered. A new difficulty is introduced. Indeed, problem (1) can not be considered as two conservation laws with Lipschitz continuous flux on each side of the line $\{x = 0\}$, because this approach seems to need the trace of the solution (see by example [KRT03, SV03, Bac04]).

Moreover, in [Tow00] and in [Tow01], it is only proved that a subsequence of the approximation function, build with the scheme, converges to an entropy solution. In [KT04], authors prove the convergence of the Lax-Friedrichs scheme without extraction of a subsequence but they still assume that g is genuinely non linear. In fact, they need g genuinely non linear to show the uniqueness of entropy solution, and the uniqueness permits to conclude that the whole sequence converges to the entropy solution. Recently, in [AV03, AJV04], the authors present some studies for a generalized problem for the following hyperbolic conservation law:

$$\begin{cases} \partial_t u + \partial_x (g(x,u)) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x)\,, \end{cases} \tag{2}$$

with

$$g(x,u) = \begin{cases} g_L(u) & \text{if} \quad x < 0, \\ g_R(u) & \text{if} \quad x > 0, \end{cases}$$

such that $g_L(0) = g_R(0) = g_R(1) = g_R(0)$. Assuming g_L and g_R convexs, an explicit formula of the solution to problem (2) is given.

To begin, the notion of entropy solution to problem (1) is recalled. Generically, the discontinuity of k enforces the instantaneous apparition of discontinuities in the solution to problem (1) (whatever the regularity of the initial value may be). In order to ensure uniqueness, weak solutions satisfying entropy inequalities have to be considered.

Definition 1.1 Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$ a.e. on $\mathbb{R}$. A function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is said to be an entropy solution to problem (1) if it satisfies the following entropy inequalities : for all $\kappa \in [0,1]$, for all non-negative function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}} |u(t,x) - \kappa| \, \partial_t \varphi(t,x) \, dt \, dx \\ + & \int_0^\infty \int_{\mathbb{R}} (k(x) \, \Phi(u(t,x),\kappa) + \Psi(u(t,x),\kappa)) \, \partial_x \varphi(t,x) \, dx \, dt \\ + & \int_{\mathbb{R}} |u_0(x) - \kappa| \, \varphi(0,x) \, dx \, + |k_L - k_R| \int_0^\infty g(\kappa) \, \varphi(t,0) \, dt \geq 0, \end{aligned} \tag{3}$$

where respectively Φ and Ψ denote the entropy flux associated with the Kruzhkov entropy:

$$\begin{aligned} \Phi(u,\kappa) &= \mathrm{sgn(u-\kappa)(g(u) - g(\kappa))}, \\ \text{and} \quad \Psi(u,\kappa) &= \mathrm{sgn(u-\kappa)(f(u) - f(\kappa))}. \end{aligned}$$

Remark 1 Let $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ be an entropy solution. Choosing $\kappa = 1$ in inequality (3) and using $g(1) = 0$, it is easy to see that $u \leq 1$ a.e.. Similarly, choosing $\kappa = 0$ in inequality (3), we obtain $u \geq 0$ a.e.. Then $0 \leq u \leq 1$ a.e.. This property will also be satisfied by the approximated solution given by the scheme (see Lemma 2.2).

This paper is divided into two main parts. First, the convergence of the scheme is established. In section 2, the scheme is presented : this scheme is Euler explicit in time and finite volume in space. Both discretizations (in time and in space) are of first order. The aim of subsection 2.3 is to establish a stability property which is verified by the approximate solution given by the scheme (see Theorem 2.3). In subsection 2.5, some discrete entropy inequalities which are satisfied by the approximated solution are established. In particular, the monotonicity of the scheme is introduced in subsection 2.2 is used.
By use of stability property of the scheme, in section 3, the existence of entropy process solution is established. This notion appears as a generalization of entropy solution. The convergence of a subsequence to an entropy process solution is proved. Finally, using the theorem of comparison between two entropy process solutions established in [BV05], the equivalence of entropy solution and entropy process solution and the uniqueness of entropy solution are deduced. Then, the convergence of the scheme to the unique entropy solution to problem (1) is obtained.

Secondly, some numerical results are presented. On the one hand, the behaviour of Godunov scheme and VFRoe-ncv scheme is studied with g neither convex nor concave. The approximated function build with this scheme converges to the entropy solution. We observe, numerically, a first order convergence.
On the other hand, problem (1), setting $f = 0$, is equivalent to the 2×2 resonant system :

$$\begin{cases} \partial_t u + \partial_x \big(k(x) g(u)\big) = 0, \\ \partial_t k = 0. \end{cases} \tag{4}$$

This system is resonant for all values where g' is equal to zero. In fact, if the function g is constant on an interval I included in $[0,1]$, system (4) is resonant on I. However, for such a function g, we show that problem (1) is well posed. Godunov, VFRoe-ncv schemes are presented. The convergence of these schemes is observed although the VFRoe-ncv scheme is not monotone. Moreover, all these schemes have the same behaviour.

2 Finite volume scheme

2.1 Presentation of the scheme

DEFINITION 2.1 An admissible mesh $\mathcal{T}$ of $\mathbb{R}$ is given by an increasing sequence of real values $(x_{i+1/2})_{i\in\mathbb{Z}}$, such that $\mathbb{R} = \bigcup_{i\in\mathbb{Z}}[x_{i-1/2}, x_{i+1/2}]$. The mesh $\mathcal{T}$ is the set of $\mathcal{T} = \{K_i,\, i \in \mathbb{Z}\}$ of subsets of $\mathbb{R}$ defined by $K_i = (x_{i-1/2}, x_{i+1/2})$ for all $i \in \mathbb{Z}$. The length of K_i is denoted by h_i, and set $h = \text{size}(\mathcal{T}) = \sup_{i\in\mathbb{Z}} \mathrm{h_i}$.

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 2.1 and let $\Delta t \in \mathbb{R}_+^*$ be the time step. To fix the notation, one assumes that $x_{1/2} = 0$.
In the general case, the finite volume scheme for the discretization of problem (1) can be written: $\forall i \in \mathbb{Z}$, $\forall n \in \mathbb{N}$

$$\begin{cases} \dfrac{h_i}{\Delta t}(u_i^{n+1} - u_i^n) + H(u_i^n, u_{i+1}^n, k_i, k_{i+1}) - H(u_{i-1}^n, u_i^n, k_{i-1}, k_i) = 0, \\[2ex] u_i^0 = \dfrac{1}{h_i}\displaystyle\int_K u_0(x)dx, \quad k_i = \dfrac{1}{h_i}\displaystyle\int_K k(x)dx, \end{cases} \tag{5}$$

where u_i^n is expected to be an approximation of u at time $t_n = n\Delta t$ in cell K_i. The quantity $H(u_i^n, u_{i+1}^n, k_i, k_{i+1})$ is the numerical flux at point $x_{i+1/2}$ and time t_n associated to the function $k(x)g(u) + f(u)$.
The formulation (5) is equivalent to:

$$u_i^{n+1} \quad = \quad G(u_{i-1}^n, u_i^n, u_{i+1}^n, k_{i-1}, k_i, k_{i+1}). \tag{6}$$

The approximated finite volume solution is defined by

$$u_{\mathcal{T},\Delta t}(x,t) = u_i^n \text{ for } x \in K_i \text{ and } t \in [nk, (n+1)k). \tag{7}$$

The flux functions satisfy the following hypotheses :

(H4) **Flux adapted to the function g:** $\forall u, v \in [0,1]$, $H(u,v,k_L,k_L) = H_L(u,v)$, $H(u,v,k_R,k_R) = H_R(u,v)$, $H(0,0,k_L,k_R) = H(1,1,k_L,k_R) = 0$ and $H_L(0,0) = H_L(1,1) = H_R(0,0) = H_R(1,1) = 0$.

(H5) **Regularity:** The function H is locally Lipschitz continuous from $\mathbb{R}^4$ to $\mathbb{R}$ and admits as Lipschitz constant $L_{k,g,f}$ only depending of k, g and f.

(H6) **Consistency** : $\forall u \in [0,1]$, $H_L(u,u) = k_L g(u) + f(u)$ and $H_R(u,u) = k_R g(u) + f(u)$.

(H7) **Monotonicity:** $(u,v,k_1,k_2) \mapsto H(u,v,k_1,k_2)$, from $[0,1]^4$ to $\mathbb{R}$, is nondecreasing with respect to u, k_1, k_2, and nonincreasing with respect to v.

2.2 Monotonicity of the scheme and L^∞ estimate

Lemma 2.2 Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 2.1 and let $\Delta t \in \mathbb{R}_+^*$ be the time step. Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$ a.e. on $\mathbb{R}$.
Let $u_{\mathcal{T},\Delta t}$ be the finite volume approximated solution defined by (7). Under the CFL condition

$$\Delta t \le \frac{\inf_{i\in\mathbb{Z}} h_i}{2L_{k,g,f}} \tag{8}$$

the function G is nondecreasing with respect to its three first arguments and the approximation $u_{\mathcal{T},\Delta t}$ satisfies

$$0 \le u_{\mathcal{T},k} \le 1 \quad \text{for a.e. } x \in \mathbb{R} \text{ and a.e. } t \in \mathbb{R}_+. \tag{9}$$

For the proof, we assume for simplicity that G is $\mathcal{C}^1$. Under the CFL condition, the partial differentials of G defined by (6) are non negative. Then, the monotonicity of function G and the following equalities : $G(0,0,0,.,.,.) = 0$ and $G(1,1,1,.,.,.) = 1$ are used.
G is nondecreasing with respect to its three first arguments :

-
$$\begin{aligned}\frac{\partial G}{\partial u_i^n} &= 1 - \frac{\Delta t}{h_i} H_u(u_i^n, u_{i+1}^n, k_i, k_{i+1}) + \frac{\Delta t}{h_i}(H_v(u_{i-1}^n, u_i^n, k_{i-1}, k_i)\\ &\ge 1 - 2\frac{\Delta t}{h_i} L_{k,g,f} \ge 0,\end{aligned}$$

  under the CFL condition.

-
$$\frac{\partial G}{\partial u_{i+1}^n} = -\frac{\Delta t}{h_i} H_v(u_i^n, u_{i+1}^n, k_i, k_{i+1}) \ge 0,$$

  because H is nonincreasing with respect to its second argument.

-
$$\frac{\partial G}{\partial u_{i-1}^n} = \frac{\Delta t}{h_i} H_u(u_{i-1}^n, u_i^n, k_{i-1}, k_i) \ge 0,$$

  because H is nondecreasing with respect to its first argument.

By hypothesis, $0 \le u_i^0 \le 1$ a.e. on $\mathbb{R}$. For all i in $\mathbb{Z}$, using monotonicity argument, we obtain :

$$\begin{aligned}G(0,0,0,k_{i-1},k_i,k_{i+1}) \le u_i^1 &= G(u_i^0, u_{i-1}^0, u_{i+1}^0, k_{i-1}, k_i, k_{i+1})\\ &\le G(1,1,1,k_{i-1},k_i,k_{i+1})\end{aligned}$$

with $G(0,0,0,k_{i-1},k_i,k_{i+1}) = 0$ and $G(1,1,1,k_{i-1},k_i,k_{i+1}) = 1$, because $g(0) = g(1) = 0$ (see (H1)).
Inequality (9) is deduced by induction on n.

2.3 Weak BV estimates

THEOREM 2.3 Let $\xi \in (0,1)$ and $\alpha \in (0,1)$ be given values. Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 2.1 such that $\alpha h \le h_i$ for all $i \in \mathbb{Z}$. Let $\Delta t \in \mathbb{R}_+^*$ satisfying the CFL condition

$$\Delta t \le \frac{(1-\xi)\alpha \inf_{i\in\mathbb{Z}} h_i}{2L_{k,g,f}}. \tag{10}$$

Let $\{u_i^n, i \in \mathbb{Z}, n \in \mathbb{N}\}$ be given by the finite volume scheme (5). Let $R \in \mathbb{R}_+^*$ and $T \in \mathbb{R}_+^*$ and assume $h < R$ and $\Delta t < T$. Let $i_0, i_2 \in \mathbb{Z}$ and $N_T \in \mathbb{N}$ such that: $-R \in \bar{K}_{i_0}$, $R \in \bar{K}_{i_2}$ and $T \in]N_T\Delta t, (N_T+1)\Delta t]$. Then there exists $C \in R_+^*$, only depending on g, f, R, T, u_0, ξ and α, such that

$$\begin{aligned}
&\sum_{n=0}^{N} \Delta t \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u\ ,u\ _{+1})} |k_L g(p) + f(p) - H_L(p,q)| \\
&\qquad + \max_{(p,q)\in\mathcal{C}(u\ ,u\ _{+1})} |k_L g(q) + f(q) - H_L(p,q)| \\
+ &\sum_{n=0}^{N} \Delta t \sum_{i=1}^{i_2} \max_{(p,q)\in\mathcal{C}(u\ ,u\ _{+1})} |k_R g(q) + f(q) - H_R(p,q)| \\
&\qquad + \max_{(p,q)\in\mathcal{C}(u\ ,u\ _{+1})} |k_R g(q) + f(q) - H_R(p,q)| \le \frac{C}{\sqrt{h}},
\end{aligned} \tag{11}$$

with for a, b real values, $\mathcal{C}(a,b) = \{(p,q) \in [a\bot b, a\top b]; (q-p)(b-a) \ge 0\}$.

To establish this estimate, some tools introduced in [EGH00] for conservation laws are used. But in this preceding work, they strongly use that k is Lipschitz continuous which is not the case in this work. First we focus our study on the left and on the right of the line $\{x = 0\}$ and next around of it.

2.4 Proof of Theorem 2.3

In order to prove (11), equality (5) is multiplied by $h_i u_i^n$ and the result is summed over $i = i_0, \ldots, -1$ or over $i = 1, \ldots, i_2$, and over $n = 0, \ldots, N_T$.

Remark 2 In this part, C_j denotes constant only depending on k, g, f, T, R, u_0, ξ, α.

On the one hand, $\boxed{\text{for } i = i_0, \ldots, -1,\ k_{i-1} = k_i = k_{i+1} = k_L}$, the sum gives:

$$B_1 + B_2 = 0$$

where

$$B_1 = \sum_{n=0}^{N} \sum_{i=i_0}^{-1} h_i (u_i^{n+1} - u_i^n) u_i^n, \tag{12}$$

$$B_2 = \sum_{n=0}^{N} \sum_{i=i_0}^{-1} \Delta t \left(H_L(u_i^n, u_{i+1}^n) - H_L(u_{i-1}^n, u_i^n) \right) u_i^n. \tag{13}$$

Each term is studied separately.

1. **Study of term** B_2
   A change of index permits to obtain:

$$\begin{aligned}
B_2 &= \sum_{n=0}^{N}\sum_{i=i_0}^{-1} \Delta t(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_i^n) + f(u_i^n)))u_i^n \\
&- \sum_{n=0}^{N}\sum_{i=i_0}^{-1} \Delta t(H_L(u_{i-1}^n,u_i^n) - (k_L g(u_i^n) + f(u_i^n)))u_i^n \\
&= \sum_{n=0}^{N}\sum_{i=i_0}^{-1} \Delta t(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_i^n) + f(u_i^n)))u_i^n \\
&- \sum_{n=0}^{N}\sum_{i=i_0-1}^{-2} \Delta t(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_{i+1}^n) + f(u_{i+1}^n)))u_{i+1}^n \\
&= \sum_{n=0}^{N}\sum_{i=i_0}^{-1} \Delta t(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_i^n) + f(u_i^n)))u_i^n \\
&\qquad -(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_{i+1}^n) + f(u_{i+1}^n)))u_{i+1}^n \\
&- \sum_{n=0}^{N} \Delta t(H_L(u_{i_0-1}^n,u_{i_0}^n) - (k_L g(u_{i_0}^n) + f(u_{i_0}^n)))u_{i_0}^n \\
&+ \sum_{n=0}^{N} \Delta t(H_L(u_{-1}^n,u_0^n) - (k_L g(u_0^n) + f(u_0^n)))u_0^n \\
&= B_2^1 + B_2^2,
\end{aligned}$$

   with

$$\begin{aligned}
B_2^1 = \sum_{n=0}^{N}\sum_{i=i_0}^{-1} \Delta t\Big(&\big(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_i^n) + f(u_i^n))\big)u_i^n \\
&-\big(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_{i+1}^n) + f(u_{i+1}^n))\big)u_{i+1}^n\Big),
\end{aligned}$$

   and

$$|B_2^2| \;\leq\; C_1.$$

   Denoting by Φ_L a primitive of the function $(.)k_L g'(.) + (.)f'(.)$, an integration by parts yields, for all a, b real values

$$\begin{aligned}
\Phi_L(b) - \Phi_L(a) &= \int_a^b s\,(k_L g'(s) + f'(s))\,ds \\
&= a\big(H_L(a,b) - (k_L g(a) + f(a))\big) \\
&- b\big(H_L(a,b) - (k_L g(b) + f(b))\big) \\
&- \int_a^b \big(k_L g(s) + f(s) - H_L(a,b)\big)\,ds.
\end{aligned}$$

Then, B_2^1 becomes :

$$
\begin{aligned}
B_2^1 &= \sum_{n=0}^{N} \Delta t \sum_{i=i_0}^{-1} \Phi_L(u_{i+1}^n) - \Phi_L(u_i^n) \\
&+ \sum_{n=0}^{N} \Delta t \sum_{i=i_0}^{-1} \int_{u}^{u\ \ +1} \big(k_L g(s) + f(s) - H_L(u_i^n, u_{i+1}^n)\big)\, ds \\
&= B_2^{1,1} + B_2^{1,2},
\end{aligned}
$$

with, immediately $|B_2^{1,1}| \leq C_2$. For study term $B_2^{1,2}$, one needs the following result:

Lemma 2.4 Let $f \in \mathcal{C}(\mathbb{R})$ and $j \in \mathcal{C}(\mathbb{R}^2)$ Lipschitz continuous which satisfies for all $s \in \mathbb{R}$ $j(s,s) = f(s)$ and which is nondecreasing with respect to its first argument and nonincreasing with respect to its second argument. Let j_1 and j_2 be the Lipschitz constants of j with respect to its two variables. Let $(a,b) \in \mathbb{R}^2$, then f and j satisfy the following inequality :

$$
\begin{aligned}
\int_a^b \big(f(s) - j(a,b)\big)\, ds \;\; &\geq \;\; \frac{1}{2(j_1 + j_2)} \Big(\max_{(p,q)\in\mathcal{C}(a,b)} \big(f(p) - j(p,q)\big)^2 \\
&\qquad + \max_{(p,q)\in\mathcal{C}(a,b)} \big(f(q) - j(p,q)\big)^2 \Big).
\end{aligned}
$$

The reader can find the proof of this lemma in the Handbook of numerical analysis [EGH00] (page 915).

Using $H_L(s,s) = k_L g(s) + f(s)$ with H_L nondecreasing with respect to its first argument and nonincreasing with respect to its second argument, and applying Lemma 2.4 to $k_L g + f$ and H_L, $B_2^{1,2}$, we get

$$
\begin{aligned}
B_2^{1,2} \;\; \geq \;\; \frac{1}{2\, L_{k,g,f}} \sum_{n=0}^{N} \Delta t \sum_{i=i_0}^{-1} &\Big(\max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ +1)} \big(k_L g(p) + f(p) - H_L(p,q)\big)^2 \\
&+ \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ +1)} \big(k_L g(q) + f(q) - H_L(p,q)\big)^2 \Big).
\end{aligned}
$$

Then, this yields

$$
\begin{aligned}
B_2 \;\; \geq \;\; \frac{1}{2\, L_{k,g,f}} \sum_{n=0}^{N} \Delta t \sum_{i=i_0}^{-1} &\Big(\max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ +1)} \big(k_L g(p) + f(p) - H_L(p,q)\big)^2 \\
&+ \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ +1)} \big(k_L g(q) + f(q) - H_L(p,q)\big)^2 \Big) \\
-(C_1 + C_2). &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (14)
\end{aligned}
$$

2. **Study of B_1**
Using the definition of B_1 (12), one has

$$
\begin{aligned}
B_1 &= -\frac{1}{2}\sum_{n=0}^{N}\sum_{i=i_0}^{-1}(u_i^{n+1}-u_i^n)^2 - \frac{1}{2}\sum_{i=i_0}^{-1}(u_i^0)^2 + \frac{1}{2}\sum_{i=i_0}^{-1}(u_i^{N\ \ +1})^2 \\
&\geq -\frac{1}{2}\sum_{n=0}^{N}\sum_{i=i_0}^{-1}(u_i^{n+1}-u_i^n)^2 - \frac{1}{2}\sum_{i=i_0}^{-1}(u_i^0)^2. \qquad (15)
\end{aligned}
$$

Using scheme (5), for $i \in \{i_0, \dots, -1\}$, with the CFL condition (10), this yields

$$
\begin{aligned}
h_i(u_i^{n+1}-u_i^n)^2 &= \frac{\Delta t^2}{h_i}\Big([H_L(u_i^n,u_{i+1}^n)-(g_L(u_i^n)+f(u_i^n))] \\
&\quad -[H_L(u_{i-1}^n,u_i^n)-(g_L(u_i^n)+f(u_i^n))]\Big)^2 \\
&\leq \frac{(1-\xi)\Delta t}{L_{k,g,f}} \\
&\quad \times\Big([H_L(u_i^n,u_{i+1}^n)-(k_L g(u_i^n)+f(u_i^n))]^2 \\
&\quad +[H_L(u_{i-1}^n,u_i^n)-(k_L g(u_i^n)+f(u_i^n))]^2\Big).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{2}\sum_{n=0}^{N}\sum_{i=i_0}^{-1} h_i(u_i^{n+1}-u_i^n)^2 \\
\leq\ & \frac{(1-\xi)}{2L_{k,g,f}}\Big(\sum_{n=0}^{N}\Delta t\sum_{i=i_0}^{-1}[H_L(u_i^n,u_{i+1}^n)-(k_L g(u_i^n)+f(u_i^n))]^2 \\
&\qquad +[H_L(u_i^n,u_{i+1}^n)-(k_L g(u_{i+1}^n)+f(u_{i+1}^n))]^2\Big) \\
&\qquad +C_5 \\
\leq\ & \frac{(1-\xi)}{2L_{k,g,f}} \\
&\times\Big(\sum_{n=0}^{N}\Delta t\sum_{i=i_0}^{-1}\max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})}[k_L g(p)+f(p)-H_L(p,q)]^2 \\
&\qquad +\max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})}[k_L g(q)+f(q)-H_L(p,q)]^2\Big) \\
&\qquad +C_5. \qquad (16)
\end{aligned}
$$

Using the preceding inequality, equation (15) gives

$$
\begin{aligned}
B_1 \quad \geq \quad & -\frac{(1-\xi)}{2L_{k,g,f}} \\
& \times\Big(\sum_{n=0}^{N} \Delta t \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})} [k_L g(p) + f(p) - H_L(p,q)]^2 \\
& \qquad + \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})} [k_L g(q) + f(q) - H_L(p,q)]^2\Big) - C_6,
\end{aligned}
\tag{17}
$$

with $C_6 = C_5 + \dfrac{1}{2}\displaystyle\sum_{i=i_0}^{-1}(u_i^0)^2$.

3. **Final estimate**

Adding (14) and (17) and using $B_1 + B_2 = 0$, this yields

$$
\begin{aligned}
0 \quad = \quad & B_1 + B_2 \\
\geq \quad & \frac{\xi}{2L_{k,g,f}} \\
& \times \sum_{n=0}^{N} \Delta t \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})} [k_L g(p) + f(p) - H_L(p,q)]^2 \\
& \qquad + \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})} [k_L g(q) + f(q) - H_L(p,q)]^2 \\
- \quad & \bar{C}_7.
\end{aligned}
$$

Then

$$
\begin{aligned}
& \sum_{n=0}^{N} \Delta t \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})} [k_L g(p) + f(p) - H_L(p,q)]^2 \\
+ & \qquad\qquad \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})} [k_L g(q) + f(q) - H_L(p,q)]^2 \leq C_7.
\end{aligned}
\tag{18}
$$

On the second hand, $\boxed{\text{for } i = 2, \ldots, i_2,\ k_{i-1} = k_i = k_{i+1} = k_R}$, in the same manner as above

$$
\begin{aligned}
& \sum_{n=0}^{N} \Delta t \sum_{i=2}^{i_2} \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})} [k_R g(p) + f(p) - H_R(p,q)]^2 \\
& \qquad + \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})} [k_R g(q) + f(q) - H_R(p,q)]^2 \leq C_8.
\end{aligned}
\tag{19}
$$

Moreover

$$
\begin{aligned}
& \sum_{n=0}^{N} \Delta t \max_{(p,q)\in\mathcal{C}(u_1,u_2)} [k_R g(p) + f(p) - H_R(p,q)]^2 \\
& \qquad + \max_{(p,q)\in\mathcal{C}(u_1,u_2)} [k_R g(q) + f(q) - H_R(p,q)]^2 \leq C_9,
\end{aligned}
\tag{20}
$$

because $\displaystyle\sum_{n=0}^{N} \Delta t \leq T$.
Finally, adding (18), (19) and (20), this yields:

$$\begin{aligned}
& \sum_{n=0}^{N} \Delta t \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})} [k_L g(p) + f(p) - H_L(p,q)]^2 \\
& \qquad + \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})} [k_L g(q) + f(q) - H_L(p,q)]^2 \\
+ \quad & \sum_{n=0}^{N} \Delta t \sum_{i=1}^{i_2} \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})} [k_R g(p) + f(p) - H_R(p,q)]^2 \\
& \qquad + \max_{(p,q)\in\mathcal{C}(u\ \ ,u\ \ _{+1})} [k_R g(q) + f(q) - H_R(p,q)]^2 \leq C_{12}.
\end{aligned}$$

To obtain estimate (11) and conclude the proof of Theorem 2.3, it is sufficient to apply the Cauchy-Schwartz inequality to the preceding inequality.

2.5 Discrete entropy inequalities

THEOREM 2.5 Under (H4) to (H7), let $\mathcal{T}$ be an admissible mesh in the sense of Definition 2.1 and $\Delta t \in \mathbb{R}_+^*$ the time step. Let $\{u_i^n, i \in \mathbb{Z}, n \in \mathbb{N}\}$ be given by (5); then for all $\kappa \in [0,1]$, $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, the following inequality holds :

$$|u_i^{n+1} - \kappa| \leq |u_i^n - \kappa| - \frac{\Delta t}{h_i}(G^n_{i+\frac{1}{2}} - G^n_{i-\frac{1}{2}}) + \frac{\Delta t}{h_i}|\Delta h^i| \tag{21}$$

with

$$\begin{aligned}
G^n_{i+\frac{1}{2}} &= H(u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_i, k_{i+1}) - H(u_i^n \perp \kappa, u_{i+1}^n \perp \kappa, k_i, k_{i+1}), \\
\text{and } |\Delta h^i| &= |H(\kappa, \kappa, k_i, k_{i+1}) - H(\kappa, \kappa, k_{i-1}, k_i)|.
\end{aligned}$$

The proof is based on the monotonicity of the scheme and on the following equality: $u \top \kappa - u \perp \kappa = |u - \kappa|$ with $u \top \kappa = max(u, \kappa)$ and $u \perp \kappa = min(u, \kappa)$.

Let $i \in \mathbb{Z}$, $n \in \mathbb{N}$, $\kappa \in [0,1]$ and $\lambda_i := \dfrac{\Delta t}{h_i}$.
The proof is divided into two steps according to the sign of Δh^i.

1. Assume that $\Delta h^i \geq 0$.
   On the one hand, by monotonicity, this yields :

$$\begin{aligned}
u_i^{n+1} - \lambda_i \, \Delta h^i &\leq u_i^{n+1} = G(u_{i-1}^n, u_i^n, u_i^n, k_{i-1}, k_i, k_{i+1}) \\
&\leq G(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_{i-1}, k_i, k_{i+1})
\end{aligned} \tag{22}$$

   and

$$\kappa - \lambda_i \, \Delta h^i \leq G(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_{i-1}, k_i, k_{i+1}), \tag{23}$$

then with (22) and (23)

$$(u_i^{n+1}-\lambda_i\,\Delta h^i)\top(\kappa-\lambda_i\,\Delta h^i)\le G(u_{i-1}^n\top\kappa,u_i^n\top\kappa,u_{i+1}^n\top\kappa,k_{i-1},k_i,k_{i+1}),$$

and

$$(u_i^{n+1}\top\kappa)\le G(u_{i-1}^n\top\kappa,u_i^n\top\kappa,u_{i+1}^n\top\kappa,k_{i-1},k_i,k_{i+1})+\lambda_i\,\Delta h^i. \tag{24}$$

On the other hand,

$$\kappa\ge\kappa-\lambda_i\,\Delta h^i\ge G(u_{i-1}^n\bot\kappa,u_i^n\bot\kappa,u_{i+1}^n\bot\kappa,k_{i-1},k_i,k_{i+1}),$$

and

$$u_i^{n+1}\ge G(u_{i-1}^n\bot\kappa,u_i^n\bot\kappa,u_{i+1}^n\bot\kappa,k_{i-1},k_i,k_{i+1}),$$

then

$$u_i^{n+1}\bot\kappa\ge G(u_{i-1}^n\bot\kappa,u_i^n\bot\kappa,u_i^n\bot\kappa,k_{i-1},k_i,k_{i+1}). \tag{25}$$

Finally, combining (24) and (25) yields :

$$\begin{aligned}
|u_i^{n+1}-\kappa| &= (u_i^{n+1}\top\kappa)-(u_i^{n+1}\bot\kappa)\\
&\le \quad G(u_{i-1}^n\top\kappa,u_i^n\top\kappa,u_{i+1}^n\top\kappa,k_{i-1},k_i,k_{i+1})\\
&\quad -G(u_{i-1}^n\bot\kappa,u_i^n\bot\kappa,u_{i+1}^n\bot\kappa,k_{i-1},k_i,k_{i+1})\\
&\quad +\lambda_i\Delta h^i\\
&\le \quad G(u_{i-1}^n\top\kappa,u_i^n\top\kappa,u_{i+1}^n\top\kappa,k_{i-1},k_i,k_{i+1})\\
&\quad -G(u_{i-1}^n\bot\kappa,u_i^n\bot\kappa,u_{i+1}^n\bot\kappa,k_{i-1},k_i,k_{i+1})\\
&\quad +\lambda_i\,|\Delta h^i|.
\end{aligned} \tag{26}$$

2. If $\Delta h^i\le 0$, in the same manner

$$\begin{aligned}
|u_i^{n+1}-\kappa| &= (u_i^{n+1}\top\kappa)-(u_i^{n+1}\bot\kappa)\\
&\le \quad G(u_{i-1}^n\top\kappa,u_i^n\top\kappa,u_{i+1}^n\top\kappa,k_{i-1},k_i,k_{i+1})\\
&\quad -G(u_{i-1}^n\bot\kappa,u_i^n\bot\kappa,u_{i+1}^n\bot\kappa,k_{i-1},k_i,k_{i+1})\\
&\quad -\lambda_i\Delta h^i\\
&\le \quad G(u_{i-1}^n\top\kappa,u_i^n\top\kappa,u_{i+1}^n\top\kappa,k_{i-1},k_i,k_{i+1})\\
&\quad -G(u_{i-1}^n\bot\kappa,u_i^n\bot\kappa,u_{i+1}^n\bot\kappa,k_{i-1},k_i,k_{i+1})\\
&\quad +\lambda_i\,|\Delta h^i|.
\end{aligned} \tag{27}$$

Eventually, this yields for all $\kappa\in[0,1]$

$$\begin{aligned}
|u_i^{n+1}-\kappa| \le &\quad G(u_{i-1}^n\top\kappa,u_i^n\top\kappa,u_{i+1}^n\top\kappa,k_{i-1},k_i,k_{i+1})\\
&-G(u_{i-1}^n\bot\kappa,u_i^n\bot\kappa,u_{i+1}^n\bot\kappa,k_{i-1},k_i,k_{i+1})\\
&+\lambda_i\,|\Delta h^i|.
\end{aligned}$$

Eventually,

$$\begin{aligned}
&G(u_{i-1}^n\top\kappa,u_i^n\top\kappa,u_{i+1}^n\top\kappa,k_{i-1},k_i,k_{i+1})-G(u_{i-1}^n\bot\kappa,u_i^n\bot\kappa,u_{i+1}^n\bot\kappa,k_{i-1},k_i,k_{i+1})\\
=\ &|u_i^n-\kappa|-\lambda_i\big(G_{i+\frac12}^n-G_{i-\frac12}^n\big).
\end{aligned}$$

Then, for all $\kappa\in[0,1]$, $i\in\mathbb{Z}$ and $n\in\mathbb{N}$

$$|u_i^{n+1}-\kappa|\le|u_i^n-\kappa|-\lambda_i\big(G_{i+\frac12}^n-G_{i-\frac12}^n\big)+\lambda_i\,|\Delta h^i|.$$

3 Entropy process solution

Now the convergence of the scheme to an entropy process solution is presented. This convergence result is obtained in the sense of "nonlinear weak-$\star$ convergence", defined in [EGH00], which is a convenient way to understand the convergence towards a Young's measure (see [DiP85]):

DEFINITION 3.1 Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 1)$, $(u_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ and $u \in L^\infty(\Omega \times (0,1))$. The sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in the *nonlinear weak-$\star$ sense* if

$$\int_\Omega h(u_n(x))\psi(x)\,dx \to \int_0^1 \int_\Omega h(u(x,\alpha))\psi(x)dx\,d\alpha, \text{ as } n \to +\infty$$
$$\forall \psi \in L^1(\Omega),\ \forall h \in \mathcal{C}(\mathbb{R},\mathbb{R}). \tag{28}$$

Equivalentlyy, the sequence $(u_n)_{n\in\mathbb{N}}$ converges to $u \in L^\infty(\Omega\times(0,1))$ in the nonlinear weak-$\star$ sense if, for every $h \in \mathcal{C}(\mathbb{R},\mathbb{R})$, the nonlinear expression $h(u_n)$ converges in $L^\infty(\Omega)$ weak-$\star$ to a limit which has the structure $\int_0^1 h(u(\cdot,\alpha))d\alpha$. The fact is, that any bounded sequence of $L^\infty(\Omega)$ has a subsequence converging in the nonlinear weak-$*$ sense :

THEOREM 3.2 Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 1)$ and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^\infty(\Omega)$. Then $(u_n)_{n\in\mathbb{N}}$ admits a subsequence converging in the nonlinear weak-$\star$ sense.

The notion of entropy process solution is adapted to problem (1) as follows :

DEFINITION 3.3 Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$ a.e. in $\mathbb{R}$. Let $u \in L^\infty(\mathbb{R}_+^* \times \mathbb{R} \times (0,1))$. The function u is an entropy process solution of problem (1) if for any $\kappa \in [0,1]$ and any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ non negative,

$$\begin{aligned}
&\int_0^\infty \int_\mathbb{R} \int_0^1 |u(t,x,\alpha) - \kappa|\,\partial_t\varphi(t,x)\,dt\,dx\,d\alpha \\
+ &\int_0^\infty \int_\mathbb{R} \int_0^1 (k(x)\,\Phi(u(t,x,\alpha),\kappa) + \Psi(u(t,x,\alpha),\kappa))\,\partial_x\varphi(t,x)\,dx\,dt\,d\alpha \\
+ &\int_\mathbb{R} |u_0(x) - \kappa|\,\varphi(0,x)\,dx \;+\; |k_L - k_R| \int_0^\infty g(\kappa)\,\varphi(t,0)\,dt \geq 0.
\end{aligned} \tag{29}$$

THEOREM 3.4 Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$ a.e. in $\mathbb{R}$. Let $\xi \in (0,1)$ and $\alpha \in (0,1)$ be given. Let $(\mathcal{T}_m)_{m\in\mathbb{N}}$ be a sequence of an admissible mesh in the sense of Definition 2.1 such that for all $m \in \mathbb{N}$, $i \in \mathbb{N}$, $\alpha\,\mathrm{size}(\mathcal{T}_\mathrm{m}) \leq \mathrm{h_i^m}$. Let $(\Delta t_m)_{m\in\mathbb{N}}$ be a sequence of real positive values satisfying the CFL condition (10).
For all $m \in \mathbb{N}$, let $u_{\mathcal{T}\ ,\Delta t}$ be the finite volume approximated solution defined by (7). Then a subsequence of $(u_{\mathcal{T}\ ,\Delta t\ })_{m\in\mathbb{N}}$ converges towards $v \in L^\infty(\mathbb{R}_+ \times\mathbb{R}\times (0,1))$ in the weak-$\star$ nonlinear sense, as $\bar{h}_m := \mathrm{size}(\mathcal{T}_\mathrm{m}) \to 0$ and v is an entropy process solution to problem (1).

3.1 Proof of Theorem 3.4

By monotonicity of the scheme and as $0 \leq u_0 \leq 1$ a.e., $|u_{\mathcal{T}\ ,\Delta t}\ | \leq 1$ for all $m \in \mathbb{N}$. Then by convergence in the non linear weak-$\star$ sense, there exists a subsequence of $(u_{\mathcal{T}\ ,\Delta t}\ )_{m\in\mathbb{N}}$ and $v \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times (0,1))$ such that this subsequence converges to v in the weak-$\star$ nonlinear sense.

To establish that v is an entropy process solution, equation (21) is multiplied by $\dfrac{1}{\Delta t}\displaystyle\int_{t}^{t\ \ +1}\int_K \varphi(t,x)\,dt\,dx$ and one sums over i and n. The new issues (compared in [EGH00]) are the study around $x = 0$ and the study of the last term given by $\displaystyle\sum_{i\in\mathbb{Z}}\sum_{n\in\mathbb{N}} |\Delta h^i| \frac{1}{h_i}\int_{t}^{t\ \ +1}\int_K \varphi(t,x)\,dt\,dx$.

Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ and $m \in \mathbb{N}$. Let $\mathcal{T}_m = \mathcal{T}$ and $\Delta t_m = \Delta t$. As $\mathrm{supp}(\varphi)$ is compact, there exists $T > 0$ and $R > 0$ such that $\mathrm{supp}\varphi \subset [0,\mathrm{T}] \times [-\mathrm{R}+\mathrm{h}, \mathrm{R}-\mathrm{h}]$. Let i_0, i_2 and N_T be as defined in Theorem 2.3.
Let $\kappa \in [0,1]$, multiplying equation (21) by $\dfrac{1}{\Delta t}\displaystyle\int_{t}^{t\ \ +1}\int_K \varphi(t,x)\,dt\,dx$, and summing over $i = i_0, \ldots, i_2$ and $n = 0, \ldots, N_T$, yields :

$$A_1 + A_2 \leq A_3.$$

Each term is studied separately.

3.1.1 Study of term A_1

$$\begin{aligned}
A_1 &= \sum_{i=i_0}^{i_2}\sum_{n=0}^{N}\Big(|u_i^{n+1} - \kappa| - |u_i^n - \kappa|\Big)\frac{1}{\Delta t}\int_{t}^{t\ \ +1}\int_K \varphi(t,x)\,dt\,dx \\
&= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N}|u_i^n - \kappa|\int_{t}^{t\ \ +1}\int_K \frac{\varphi(t+\Delta t,x) - \varphi(t,x)}{\Delta t}\,dt\,dx \\
&\quad -\sum_{i=i_0}^{i_2}|u_i^0 - \kappa|\frac{1}{\Delta t}\int_0^k\int_K \varphi(t,x)\,dt\,dx \\
&= B_1 + B_2.
\end{aligned} \tag{30}$$

In fact, for this term, the convergence of $u_{\mathcal{T},\Delta t}$ to v for the weak-$\star$ non linear convergence as h tends to zero is used.
On the one hand

$$\begin{aligned}
B_2 &= -\sum_{i=i_0}^{i_2}|u_i^0 - \kappa|\frac{1}{\Delta t}\int_0^{\Delta t}\int_K \varphi(t,x)\,dt\,dx \\
&= -\frac{1}{\Delta t}\int_0^{\Delta t}\int_{-R}^{R}|u_{\mathcal{T},0} - \kappa|\varphi(t,x)\,dt\,dx,
\end{aligned} \tag{31}$$

with $u_{\mathcal{T},0} = \sum_{i\in\mathbb{Z}} u_i^0 1_K$.

However $u_{\mathcal{T},0}$ converges towards u_0 in $L^1_{loc}(\mathbb{R})$ and $\dfrac{1}{\Delta t}\displaystyle\int_0^{\Delta t} \varphi(t,x)\,dt$ converges towards $\varphi(0,x)$ as size$(\mathcal{T})$ tends to zero. This yields

$$B_2 \quad \to \quad \int_{\mathbb{R}} |u_0(x) - \kappa|\varphi(0,x)\,dt\,dx, \quad \text{as } h \text{ tends to zero.},$$

On the other hand,

$$\begin{aligned} B_1 &= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N} |u_i^n - \kappa| \int_t^{t\ +1} \int_K \frac{\varphi(t+\Delta t, x) - \varphi(t,x)}{\Delta t}\,dt\,dx \\ &= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N} \int_t^{t\ +1} \int_K |u_{\mathcal{T},k}(t,x) - \kappa| \frac{\varphi(t+\Delta t, x) - \varphi(t,x)}{\Delta t}\,dt\,dx \\ &= -\int_0^T \int_{-R}^{R} |u_{\mathcal{T},k}(t,x) - \kappa| \frac{\varphi(t+\Delta t, x) - \varphi(t,x)}{\Delta t}\,dt\,dx. \end{aligned}$$

$u_{\mathcal{T},k}$ converges towards v in the nonlinear weak-$\star$ sense as $h \to 0$, then

$$\int_0^T \int_{-R}^{R} |u_{\mathcal{T},k}(t,x) - \kappa|\,dt\,dx \underset{h\to 0}{\to} \int_0^T \int_{-R}^{R} \int_0^1 |v(t,x,\alpha) - \kappa|\,dt\,dx.$$

and by use of the regularity of the function φ

$$\frac{\varphi(t+\Delta t, x) - \varphi(t,x)}{\Delta t} \underset{h\to 0}{\to} \partial_t \varphi(t,x).$$

then

$$B_1 \underset{h\to 0}{\to} -\int_0^T \int_{-R}^{R} \int_0^1 |v(t,x,\alpha) - \kappa| \partial_t\varphi(t,x)\,dt\,dx.$$

One concludes

$$\begin{aligned} \lim_{h\to 0} A_1 &= -\int_0^T \int_{\mathbb{R}} \int_0^1 |v(t,x,\alpha) - \kappa| \partial_t\varphi(t,x)\,dt\,dx\,d\alpha \\ &\quad -\int_0^T |u_0(x) - \kappa|\varphi(0,x)\,dx. \end{aligned} \tag{32}$$

3.1.2 Study of term A_2

Term A_2 is defined by:

$$\begin{aligned} A_2 &= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N} \frac{1}{h_i}(G^n_{i+1/2} - G^n_{i-1/2}) \int_t^{t\ +1} \int_K \varphi(t,x)\,dt\,dx \\ &= -\sum_{i=i_0+1}^{i_2-1}\sum_{n=0}^{N} \frac{1}{h_i}(G^n_{i+1/2} - G^n_{i-1/2}) \int_t^{t\ +1} \int_K \varphi(t,x)\,dt\,dx, \end{aligned} \tag{33}$$

because $supp(\varphi) \subset [-R+h, R-h]$.

This term A_2 is new compared with a conservation law with Lipschitz continuous flux function. The discontinuity of the function k introduces new difficulties. Then, several steps are needed to establish that

$$\lim_{h\to 0} A_2 = -\int_0^1 \int_0^\infty \int_{\mathbb{R}} \big(k(x)\Phi(v(t,x,\alpha),\kappa) + \Psi(v(t,x,\alpha),\kappa)\big)\,\partial_x\varphi(t,x)\,dx\,dt\,d\alpha.$$

- At first

$$\lim_{h\to 0} |A_2 - A_{20}| = 0 \tag{34}$$

with A_{20} defined as follows:

$$\begin{aligned} A_{20} &= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N} G^n_{i+1/2}\int_{t}^{t\ \ +1}\int_{K} \partial_x\varphi(t,x)\,dt\,dx \\ &= -\sum_{i=i_0+1}^{i_2-1}\sum_{n=0}^{N}\big(G^n_{i+1/2}-G^n_{i-1/2}\big)\int_{t}^{t\ \ +1}\varphi(t,x_{i+1/2})\,dt. \end{aligned}$$

The difference between these terms is majored as follows:

$$\begin{aligned} &|A_2 - A_{20}| \\ \leq\ & \sum_{i=i_0+1}^{i_2-1}\sum_{n=0}^{N}\big|G^n_{i+1/2}-G^n_{i-1/2}\big| \\ &\qquad\qquad \int_{t}^{t\ \ +1}\Big(\Big|\varphi(t,x_{i+1/2}) - \frac{1}{h_i}\int_{K}\varphi(t,x)\,dx\Big|\Big)\,dt \\ \leq\ & \sum_{i=i_0+1}^{i_2-1}\sum_{n=0}^{N}\big|G^n_{i+1/2}-G^n_{i-1/2}\big| \\ &\qquad\qquad \Big(\int_{t}^{t\ \ +1}\frac{1}{h_i}\int_{K}\big|\varphi(t,x_{i+1/2})-\varphi(t,x)\big|\,dx\Big)\,dt \\ \leq\ & \sum_{i=i_0+1}^{i_2-1}\sum_{n=0}^{N}\big|G^n_{i+1/2}-G^n_{i-1/2}\big|\mathrm{Lip}(\varphi)\,\Delta t\,h \\ \leq\ & \mathrm{Lip}(\varphi)\,h\Big(\sum_{i=i_0+1}^{-2}\sum_{n=0}^{N}\Delta t\,\big|G^n_{i+1/2}-G^n_{i-1/2}\big| \\ &\qquad\qquad + \sum_{2}^{i_2-1}\sum_{n=0}^{N}\Delta t\,\big|G^n_{i+1/2}-G^n_{i-1/2}\big|\Big) \\ +\ & \mathrm{Lip}(\varphi)\,h\sum_{i=-1}^{1}\sum_{n=0}^{N}\Delta t\,\big|G^n_{i+1/2}-G^n_{i-1/2}\big|. \end{aligned} \tag{35}$$

$*$ For $i = i_0, \cdots, -2$, $k_i = k_{i+1} = k_L$ and

$$\begin{aligned} |G^n_{i+1/2} - G^n_{i-1/2}| &\leq |H_L(u^n_i \top \kappa, u^n_{i+1} \top \kappa) - (k_L g(u^n_i \top \kappa) + f(u^n_i \top \kappa))| \\ &+ |H_L(u^n_i \perp \kappa, u^n_{i+1} \perp \kappa) - (k_L g(u^n_i \perp \kappa) + f(u^n_i \perp \kappa))| \\ &+ |H_L(u^n_{i-1} \top \kappa, u^n_i \top \kappa) - (k_L g(u^n_i \top \kappa) + f(u^n_i \top \kappa))| \\ &+ |H_L(u^n_{i-1} \perp \kappa, u^n_i \perp \kappa) - (k_L g(u^n_i \perp \kappa) + f(u^n_i \perp \kappa))|. \end{aligned}$$

then

$$\begin{aligned} &\sum_{i=i_0+1}^{-2} \sum_{n=0}^{N} k|G^n_{i+1/2} - G^n_{i-1/2}| \\ \leq\ & 2\sum_{i=i_0}^{-1} \sum_{n=0}^{N} k\big(|H_L(u^n_i, u^n_{i+1}) - (g_L(u^n_i) + f(u^n_i))| \\ &\qquad + |H_L(u^n_i, u^n_{i+1}) - (k_L g(u^n_{i+1}) + f(u^n_{i+1}))|\big) \\ \leq\ & 2\sum_{i=i_0}^{-1} \sum_{n=0}^{N} k\big(\max_{(p,q)\in\mathcal{C}(u\ ,u\ _{+1})} |k_L g(p) + f(p) - H_L(p,q)| \\ &\qquad + \max_{(p,q)\in\mathcal{C}(u\ ,u\ _{+1})} |k_L g(q) + f(q) - H_L(p,q)|\big) \\ \leq\ & 2C\frac{1}{\sqrt{h}} \end{aligned} \tag{36}$$

using the weak-BV estimate (11).

$*$ For $i = 2, \cdots, i_2$, $k_{i-1} = k_i = k_{i+1} = k_R$. In the same manner as above

$$\begin{aligned} &\sum_{i=2}^{i_2-1} \sum_{n=0}^{N} \Delta t|G^n_{i+1/2} - G^n_{i-1/2}| \\ \leq\ & 2\sum_{i=1}^{i_2} \sum_{n=0}^{N} \Delta t\big(|H_R(u^n_i, u^n_{i+1}) - (k_R g(u^n_i) + f(u^n_i))| \\ &\qquad + |H_R(u^n_i, u^n_{i+1}) - (k_R g(u^n_{i+1}) + f(u^n_{i+1}))|\big) \\ \leq\ & 2\sum_{i=1}^{i_2} \sum_{n=0}^{N} \Delta t\big(\max_{(p,q)\in\mathcal{C}(u\ ,u\ _{+1})} |k_R g(p) + f(p) - H_R(p,q)| \\ &\qquad + \max_{(p,q)\in\mathcal{C}(u\ ,u\ _{+1})} |k_R g(q) + f(q) - H_R(p,q)|\big) \\ \leq\ & 2C\frac{1}{\sqrt{h}}. \end{aligned} \tag{37}$$

$*$ We can notice that

$$\sum_{i=-1}^{1} \sum_{n=0}^{N} \Delta t \left|G^n_{i+1/2} - G^n_{i-1/2}\right| \leq C \sum_{n=0}^{N} \Delta t \ \leq CT \tag{38}$$

$*$ Finally, with (36), (37) and (38), inequality (35) becomes:

$$|A_2 - A_{20}| \leq C\sqrt{h} \longrightarrow 0, \text{ as } h \to 0.$$

• Now, we will prove that

$$\lim_{h\to 0} |A_{20} - \bar{A}_{20}| = 0 \tag{39}$$

with $\bar{A}_{20}$ defined as follows:

$$\begin{aligned}
\bar{A}_{20} &:= -\int_0^t \int_{\mathbb{R}} \big(k(x)\Phi(v,\kappa) + \Psi(v,\kappa)\big)\partial_x\varphi(t,x)\,dt\,dx \\
&= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N}\int_t^{t\quad +1}\int_K \big(k(x)\Phi(v,\kappa) + \Psi(v,\kappa)\big)\partial_x\varphi(t,x)\,dt\,dx
\end{aligned}$$

To prove this equality one has to take into account the different value of i.
$*$ For $i = i_0, \cdots, -1$, one has $k_{i-1} = k_i = k_L$,
$*$ For $i = 1, \cdots, i_2$, one has $k_{i-1} = k_i = k_R$,
So $A_{20} = A_{20}^1 + A_{20}^2 + A_{20}^3$ and $\bar{A}_{20} = \bar{A}_{20}^1 + \bar{A}_{20}^2 + \bar{A}_{20}^3$ with

$$\begin{aligned}
A_{20}^1 &= -\sum_{i=i_0}^{-1}\sum_{n=0}^{N} G_{i+\frac{1}{2}}^n \int_t^{t\quad +1}\int_K \partial_x\varphi(t,x)\,dt\,dx \\
&= -\sum_{i=i_0}^{-1}\sum_{n=0}^{N}\int_t^{t\quad +1}\int_K \big(H_L(u_i^n \top \kappa, u_{i+1}^n \top \kappa) \\
&\qquad\qquad -H_L(u_i^n \perp \kappa, u_{i+1}^n \perp \kappa)\big)\partial_x\varphi(t,x)\,dt\,dx, \\
A_{20}^2 &= -\sum_{n=0}^{N}\int_t^{t\quad +1}\int_{K_0} \big(H(u_0^n \top \kappa, u_1^n \top \kappa, k_L, k_R) - H(u_0^n \perp \kappa, u_1^n \perp \kappa, k_L, k_R)\big) \\
&\qquad\qquad \partial_x\varphi(t,x)\,dt\,dx, \\
A_{20}^3 &= -\sum_{i=1}^{i_2}\sum_{n=0}^{N} G_{i+\frac{1}{2}}^n \int_t^{t\quad +1}\int_K \partial_x\varphi(t,x)\,dt\,dx \\
&= -\sum_{i=1}^{i_2}\sum_{n=0}^{N}\int_t^{t\quad +1}\int_K \big(H_R(u_i^n \top \kappa, u_{i+1}^n \top \kappa) \\
&\qquad\qquad -H_R(u_i^n \perp \kappa, u_{i+1}^n \perp \kappa)\big)\partial_x\varphi(t,x)\,dt\,dx,
\end{aligned}$$

and

$$
\begin{aligned}
\bar{A}_{20}^{1} &= -\sum_{i=i_0}^{-1}\sum_{n=0}^{N}\int_{t}^{t\quad +1}\int_{K}\int_{0}^{1}\big(g_L(v\top\kappa)-g_L(v\bot\kappa) \\
&\qquad\qquad +f(v\top\kappa)-f(v\bot\kappa)\big)\,\partial_x\varphi(t,x)\,dt\,dx\,d\alpha, \\
\bar{A}_{20}^{2} &= -\sum_{n=0}^{N}\int_{t}^{t\quad +1}\int_{K_0}\int_{0}^{1}\big(k(x)\Phi(v,\kappa)+\Psi(v,\kappa)\big)\partial_x\varphi(t,x)\,dt\,dx d\alpha, \\
\bar{A}_{20}^{3} &= -\sum_{i=1}^{i_2}\sum_{n=0}^{N}\int_{t}^{t\quad +1}\int_{K}\int_{0}^{1}\big(g_R(v\top\kappa)-g_R(v\bot\kappa) \\
&\qquad\qquad +f(v\top\kappa)-f(v\bot\kappa)\big)\,\partial_x\varphi(t,x)\,dt\,dx\,d\alpha.
\end{aligned}
$$

$*$ At first, the difference $A_{20}^{1}-\bar{A}_{20}^{1}$ is studied :

$$
\begin{aligned}
&|A_{20}^{1}-\bar{A}_{20}^{1}| \le \\
&\quad \sum_{i=i_0}^{-1}\sum_{n=0}^{N}\int_{t}^{t\quad +1}\int_{K}\int_{0}^{1}\Big|\big(H_L(u_i^n\top\kappa,u_{i+1}^n\top\kappa)-H_L(u_i^n\bot\kappa,u_{i+1}^n\bot\kappa)\big) \\
&\quad -\big(g_L(v\top\kappa)-g_L(v\bot\kappa)+f(v\top\kappa)-f(v\bot\kappa)\big)\Big||\partial_x\varphi(t,x)|\,dt\,dx\,d\alpha
\end{aligned}
$$

We can notice that

$$
\begin{aligned}
&\Big|\big(H_L(u_i^n\top\kappa,u_{i+1}^n\top\kappa)-H_L(u_i^n\bot\kappa,u_{i+1}^n\bot\kappa)\big) \\
&-\big(g_L(v\top\kappa)-k_Lg(v\bot\kappa)+f(v\top\kappa)-f(v\bot\kappa)\big)\Big| \\
\le\quad &\big|H_L(u_i^n\top\kappa,u_{i+1}^n\top\kappa)-(k_Lg+f)(u_i^n\top\kappa)\big| \\
&+\big|(g_L+f)(u_i^n\top\kappa)-(k_Lg+f)(v\top\kappa)\big| \\
&+\big|H_L(u_i^n\bot\kappa,u_{i+1}^n\bot\kappa)-(k_Lg+f)(u_i^n\bot\kappa)\big| \\
&+\big|(g_L+f)(u_i^n\bot\kappa)-(k_Lg+f)(v\bot\kappa)\big|.
\end{aligned}
\tag{40}
$$

Moreover, an individually study shows

$$
\begin{aligned}
&|H_L(u_i^n\top\kappa,u_{i+1}^n\top\kappa)-H_L(u_i^n\top\kappa,u_i^n\top\kappa)| \le \\
&\qquad \max_{(p,q)\in\mathcal{C}(u\quad,u\quad_{+1})}|k_Lg(p)+f(p)-H_L(p,q)|,
\end{aligned}
$$

and

$$
\begin{aligned}
&|H_L(u_i^n\bot\kappa,u_{i+1}^n\bot\kappa)-H_L(u_i^n\bot\kappa,u_i^n\bot\kappa)| \le \\
&\qquad \max_{(p,q)\in\mathcal{C}(u\quad,u\quad_{+1})}|k_Lg(p)+f(p)-H_L(p,q)|.
\end{aligned}
$$

Then, equation (40) becomes:

$$
\begin{aligned}
& \Big| \Big(H_L(u_i^n \top \kappa, u_{i+1}^n \top \kappa) - H_L(u_i^n \perp \kappa, u_{i+1}^n \perp \kappa) \Big) \\
& - \Big(g_L(v \top \kappa) - k_L g(v \perp \kappa) + f(v \top \kappa) - f(v \perp \kappa) \Big) \Big| \\
\leq \;& 2 \max_{(p,q) \in \mathcal{C}(u \;\; ,u \;\; _{+1})} |k_L g(p) + f(p) - H_L(p,q)| \\
& + (\max\{k_L, k_R\} \mathrm{Lip}(g) + \mathrm{Lip}(f)) \big| (u_i^n \top \kappa) - (v \top \kappa) \big| \\
& + (\max\{k_L, k_R\} \mathrm{Lip}(g) + \mathrm{Lip}(f)) \big| (u_i^n \perp \kappa) - (v \perp \kappa) \big| \\
\leq \;& 2 \max_{(p,q) \in \mathcal{C}(u \;\; ,u \;\; _{+1})} |k_L g(p) + f(p) - H_L(p,q)| \\
& + 2 \, (\max\{k_L, k_R\} \mathrm{Lip}(g) + \mathrm{Lip}(f)) \big| u_i^n - v \big|.
\end{aligned}
$$

Finally

$$
\begin{aligned}
& \big| A_{20}^1 - \bar{A}_{20}^1 \big| \\
\leq \;& 2 ||\partial_x \varphi||_\infty \sum_{i=i_0}^{-1} \sum_{n=0}^{N} \Delta t \, h_i \max_{(p,q) \in \mathcal{C}(u \;\; ,u \;\; _{+1})} |k_L g(p) + f(p) - H_L(p,q)| \\
+ \;& 2 ||\partial_x \varphi||_\infty \sum_{i=i_0}^{-1} \sum_{n=0}^{N} (\max\{\overset{u}{k_L}, k_R\} \mathrm{Lip}(g) + \mathrm{Lip}(f)) \\
& \qquad \int_{t}^{t \;\; +1} \int_K \int_0^1 \big| u_i^n - v(t,x,\alpha) \big| \, dt \, dx \, d\alpha \\
\leq \;& 2h ||\partial_x \varphi||_\infty \sum_{i=i_0}^{-1} \sum_{n=0}^{N} \Delta t \; \max_{(p,q}
\end{aligned}
$$

$*$ It remains to study the limit of A^2_{20} and $\bar{A}^2_{20}$.

$$\begin{aligned}
|A^2_{20}| &\leq \sum_{n=0}^{N} \int_{t}^{t\ \ +1} \int_{K_0} \left| H(u^n_0 \top \kappa, u^n_1 \top \kappa, k_L, k_R) - H(u^n_0 \perp \kappa, u^n_1 \perp \kappa, k_L, k_R) \right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad |\partial_x \varphi(t,x)| \, dt \, dx \\
&\leq C ||\partial_x \varphi||_\infty \sum_{n=0}^{N} \int_{t}^{t\ \ +1} \int_{K_0} dt \, dx \\
&\leq C ||\partial_x \varphi||_\infty T h_0 \\
&\leq C ||\partial_x \varphi||_\infty T h \longrightarrow 0, \text{ as } h \to 0,
\end{aligned}$$

and

$$\begin{aligned}
|\bar{A}^2_{20}| &\leq \sum_{n=0}^{N} \int_{t}^{t\ \ +1} \int_{K_0} \int_0^1 |(k(x)\Phi(v,\kappa) + \Psi(v,\kappa))\, \partial_x \varphi(t,x)| \, dt \, dx d\alpha \\
&\leq C ||\partial_x \varphi||_\infty \sum_{n=0}^{N} \int_{t}^{t\ \ +1} \int_{K_0} \int_0^1 dt \, dx d\alpha \\
&\leq C ||\partial_x \varphi||_\infty T h \longrightarrow 0, \text{ as } h \to 0.
\end{aligned}$$

To conclude, equality (39) had been shown.

• With (34) and (39), we obtain

$$\lim_{h \to 0} A_2 = - \int_0^t \int_{\mathbb{R}} (k(x)\Phi(v,\kappa) + \Psi(v,\kappa))\, \partial_x \varphi(t,x) \, dt \, dx. \tag{41}$$

3.1.3 Study of term A_3

Term A_3 is defined by

$$A_3 = \sum_{i=i_0}^{i_2} \sum_{n=0}^{N} |\Delta h^i| \int_{t}^{t\ \ +1} \frac{1}{h_i} \int_{K} \varphi(t,x) \, dt \, dx. \tag{42}$$

To find its limit, A_3 is divided it into three terms according to values of i.

1. For $i \in \{i_0, \ldots, -1\}$, $\Delta h^i = H(\kappa, \kappa, k_L, k_L) - H(\kappa, \kappa, k_L, k_L) = 0$,
2. For $i \in \{2, \ldots, i_2\}$, $\Delta h^i = H(\kappa, \kappa, k_R, k_R) - H(\kappa, \kappa, k_R, k_R) = 0$,
3. $|\Delta h^0| = |H(\kappa, \kappa, k_L, k_R) - H(\kappa, \kappa, k_L, k_L)| = |H(\kappa, \kappa, k_L, k_R) - k_L g(\kappa) + f(\kappa)|$, and $|\Delta h^1| = |H(\kappa, \kappa, k_R, k_R) - H(\kappa, \kappa, k_L, k_R)| = |k_R g(\kappa) + f(\kappa) - H(\kappa, \kappa, k_L, k_R)|$.

Assuming $k_L > k_R$, (it is similar if $k_L < k_R$), with hypothesis (H4)

$$\begin{aligned}
|\Delta h^0| &= (k_L g(\kappa) + f(\kappa)) - H(\kappa, \kappa, k_L, k_R) \\
\text{and } |\Delta h^1| &= H(\kappa, \kappa, k_L, k_R) - (k_R g(\kappa) + f(\kappa)).
\end{aligned}$$

Moreover

$$
\begin{aligned}
& |\Delta h^0| \int_{K_0} \varphi(t,x) dx + |\Delta h^1| \int_{K_1} \varphi(t,x) dx \\
= \quad & g(\kappa)\Big(k_L \int_{x_{-1\;2}}^{0} \varphi(t,0)\, dx - k_R \int_0^{x_{3\;2}} \varphi(t,0) dx \Big) \\
& + (H(\kappa,\kappa,k_L,k_R) - f(\kappa)) \Big(\int_0^{x_{3\;2}} \varphi(t,0)\, dx - \int_{x_{-1\;2}}^{0} \varphi(t,0)\, dx \Big) \\
\to \quad & g(\kappa)(k_L - k_R)\varphi(t,0), \quad \text{as } h \text{ tends to } 0.
\end{aligned}
$$

Finally

$$
\begin{aligned}
\lim_{h\to 0} A_3 \quad &= \quad (k_L - k_R) g(\kappa) \int_0^{+\infty} \varphi(t,0)\, dt \\
&= \quad |k_L - k_R| g(\kappa) \int_0^{+\infty} \varphi(t,0)\, dt. \qquad (43)
\end{aligned}
$$

3.1.4 Final estimate

Using $A_1 + A_2 \leq A_3$ and the limits established in previous sections (see equations (32), (41) and (43)), the function v satisfies the following inequality: for all $\kappa \in [0,1]$, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$

$$
\begin{aligned}
& \int_0^1 \int_0^\infty \int_{\mathbb{R}} |v(t,x,\alpha) - \kappa|\, \partial_t \varphi(t,x)\, d\alpha\, dt\, dx \\
+ \quad & \int_0^1 \int_0^\infty \int_{\mathbb{R}} (k(x)\Phi(v(t,x,\alpha),\kappa) + \Psi(v(t,x,\alpha),\kappa))\, \partial_x \varphi(t,x)\, d\alpha\, dx\, dt \\
+ \quad & \int_{\mathbb{R}} |u_0(x) - \kappa|\, \varphi(0,x)\, dx \; + \int_0^\infty |k_L - k_R| g(\kappa) \varphi(t,0)\, dt \geq 0.
\end{aligned}
$$

So the function $v \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times [0,1])$ is a weak entropy process solution to problem (1).

4 Convergence of the scheme

THEOREM 4.1 Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$ a.e. in $\mathbb{R}$. Let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ the unique entropy solution to problem (1). Let $\xi \in (0,1)$ and $\alpha \in (0,1)$ be given. Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 2.1 such that $\alpha h \leq h_i$ for all $i \in \mathbb{Z}$. Let $\Delta t > 0$ satisfying the CFL condition (10).
Let $u_{\mathcal{T},\Delta t}$ be the finite volume approximated solution defined by (7). Then $u_{\mathcal{T},\Delta t} \to u$ in $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R})$ for all $1 \leq p < \infty$ (and in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ for the weak-$\star$ topology), as $h = \mathrm{size}(\mathcal{T}) \to 0$.

To establish this result, a theorem of comparison between two entropy process solutions is used. This comparison is obtained in a previous work [BV05] :

THEOREM 4.2 (COMPARISON) Under hypotheses (H1), (H2), (H3), let u and $v \in L^\infty(Q \times (0,1))$ be entropy process solutions of problem (1), associated to the initial conditions $u_0 \in L^\infty(\mathbb{R}; [0,1])$ (resp. $v_0 \in L^\infty(\mathbb{R}; [0,1])$). Then, with $R, T > 0$

$$\int_0^1 \int_0^1 \int_0^T \int_{-R}^{R} (u(t,x,\lambda) - v(t,x,\zeta))^+ dx dt d\lambda d\zeta \leq T \int_{-R-CT}^{R+CT} (u_0(x) - v_0(x))^+ dx, \tag{44}$$

where $C := \max\{k_R, k_L\} \mathrm{Lip}(g) + \mathrm{Lip}(f)$.

COROLLARY 4.3 If u and $v \in L^\infty(Q \times (0,1))$ are entropy process solutions of problem (1), then $u(t,x,\lambda) = v(t,x,\zeta)$ for a.e. $(t,x,\lambda,\zeta) \in Q \times (0,1) \times (0,1)$. So $u = v$ is a classical entropy solution.

Proof of Theorem 4.1:
Let $(\mathcal{T}_m)_{m\in\mathbb{N}}$ a sequence of admissible mesh and $(\Delta t_m)_{m\in\mathbb{N}}$ a sequence of real positive values such that for all m, Δt_m satisfies the CFL condition (10). We assume that $\mathrm{size}(\mathcal{T}_\mathrm{m}) = \mathrm{h}^\mathrm{m} \to 0$.
Using Theorem 3.4 ans Corollary 4.3, a subsequence of $(u_{\mathcal{T}\ ,\Delta t}\ )_{m\in\mathbb{N}}$ converges towards an entropy process solution. Using Theorem 4.2, the entropy process solution is unique and is the entropy solution to problem (1). Then the subsequence converges towards the unique entropy solution to problem (1). Finally, as the sequence has a unique value of adherence, the whole sequence $(u_{\mathcal{T}\ ,\Delta t}\ )_{m\in\mathbb{N}}$ converges towards the entropy solution to problem (1) for the weak-$\star$ non linear topology.
Then

$$\int_0^\infty \int_{\mathbb{R}} h(u_{\mathcal{T}\ ,\Delta t}\ (t,x))\psi(t,x)\, dx\, dt \to \int_0^\infty \int_{\mathbb{R}} h(u(t,x))\psi(t,x)\, dx\, dt$$
$$\forall \psi \in L^1(\mathbb{R}^+ \times \mathbb{R}), \quad \forall h \in \mathcal{C}(\mathbb{R}, \mathbb{R}). \tag{45}$$

Setting $h(s) = s^2$ in (45) and then $h(s) = s$ and ψu instead of ψ in (45) one obtains:

$$\int_0^\infty \int_{\mathbb{R}} \big(u_{\mathcal{T}\ ,\Delta t}\ (t,x) - u(t,x)\big)^2 \psi(t,x)\, dx\, dt \to 0, \text{ as } m \to \infty,$$

for any function $\psi \in L^1(\mathbb{R}_+ \times \mathbb{R})$. From equation (45), and thanks to the L^∞ boundedness of $(u_{\mathcal{T}\ ,\Delta t}\ )_{m\in\mathbb{N}}$, the sequence $(u_{\mathcal{T}\ ,\Delta t})_{m\in\mathbb{N}}$ converges to u in $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R})$ for all $p \in [1, \infty[$.

5 Numerical methods

All the methods presented in this section are Finite Volume methods (see [EGH00]) for the hyperbolic equation (1) (with f is equal to zero, to simplify because the discontinuity of the flux doesn't concern the flux f), as scheme (5) presented in section 2.
For the sake of the simplicity, the presentation is restricted to uniform meshes (all methods may be naturally extended to non-uniform meshes). Let h be the space

step, with $h = x_{i+1/2} - x_{i-1/2}$, $i \in \mathbb{Z}$, and let Δt be the time step, with $\Delta t = t^{n+1} - t^n$, $n \in \mathbb{N}$. Besides, let u_i^n denote the approximation of $\frac{1}{\Delta x}\int_{x_{\ -1\ 2}}^{x_{\ +1\ 2}} u(t^n, x)\,dx$.
Integrating equation (1) over the cell $]x_{i-1/2}, x_{i+1/2}[\times[t^n, t^{n+1}[$ yields:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{h}\big(\varphi_{i+1/2}^n - \varphi_{i-1/2}^n\big)$$

where $\varphi_{i+1/2}^n$ is the numerical flux through the interface $\{x_{i+1/2}\} \times [t^n, t^{n+1}[$. We recall that the function k is approximated by a piecewise constant function. The numerical flux $\varphi_{i+1/2}^n$ depends on k_i, k_{i+1}, u_i^n, u_{i+1}^n.
Moreover, the CFL condition imposed in Theorem of convergence 3.4 is satisfied.

Notice that all the methods presented here rely on conservative schemes, since the problem is conservative. Finally, all the presented schemes are three-points schemes.

5.1 The Godunov scheme

The Godunov scheme [God59] is based on the resolution of the Riemann problem at each interface of the mesh. In fact we remark that problem (1), assuming $f = 0$, can be considered as the following resonant problem:

$$\partial_t u + \partial_x\big(k(x)g(u)\big) = 0, \quad \partial_t = 0. \tag{46}$$

The Godunov Method applied to this resonant system had been studied by Lin, Temple and Wang ([LTW95a], [LTW95b]). A specific Godunov scheme associated to problem (1) had been studied by Towers using a discretization of k staggered with respect to u ([Tow00], [Tow01]). Here, we consider the Godunov Method applied to the following 2×2 system:

$$\left\{\begin{array}{l} \partial_t u + \partial_x\big(k(x)g(u)\big) = 0, \\ \partial_t k = 0, \qquad\qquad t > t^n,\ x \in \mathbb{R}, \\ \\ u(0,x) = \left\{\begin{array}{ll} u_i^n & \text{if } x < x_{i+1/2} \\ u_{i+1}^n & \text{if } x > x_{i+1/2} \end{array}\right., \quad k(x) = \left\{\begin{array}{ll} k_i & \text{if } x < x_{i+1/2} \\ k_{i+1} & \text{if } x > x_{i+1/2} \end{array}\right. . \end{array}\right.$$

Let $u_{i+1/2}^n\big((x - x_{i+1/2})/(t - t^n); k_i, k_{i+1}, u_i^n, u_{i+1}^n\big)$ be the exact solution to the Riemann problem (see section 8 for an explicit presentation of the solution). Since the function k is discontinuous through the interface $\{x_{i+1/2}\} \times [t^n, t^{n+1}[$, the solution $u_{i+1/2}^n$ is discontinuous through this interface too. However, the problem is conservative, so the flux function is continuous through this interface, and writes:

$$\begin{aligned} \varphi_{i+1/2}^n &= k_i g(u_{i+1/2}^n(0^-; k_i, k_{i+1}, u_i^n, u_{i+1}^n)) \\ &= k_{i+1} g(u_{i+1/2}^n(0^+; k_i, k_{i+1}, u_i^n, u_{i+1}^n)). \end{aligned} \tag{47}$$

Remark 3 To evaluate the numerical flux $\varphi_{i+1/2}^n$, we don't have to calculate the exact solution $u_{i+1/2}^n$ but only this value at $x = 0^-$ or at $x = 0^+$. As we remark in the section 8, it is simpler.

Remark 4 In the examples presented in section 6 and 7, we can show that the Godunov scheme is monotone.

5.2 The VFRoe-ncv scheme

If we don't want to solve the Riemann problem at each step of the scheme, an alternative scheme is presented. This scheme is an approximate Godunov scheme, based on the exact solution to a linearized Riemann problem. A VFRoe-ncv scheme is defined by a change of variables (see [BGH00] and [GHS02]). The new variable is denoted by $\theta(k,u)$. For problem (1), we take $\theta(k,u) = kg(u)$ for the new variable. If v is defined by $v(t,x) = \theta(k(x), u(t,x))$, the VFRoe-ncv scheme is based on the exact resolution of the following linearized Riemann problem:

$$\left\{ \begin{array}{ll} \partial_t v + \big(\hat{k}\, g'(\hat{u})\big)\partial_x v = 0, & t > t^n,\, x \in \mathbb{R}, \\ \\ v(0,x) = \left\{ \begin{array}{ll} \theta(k_i, u_i^n) & \text{if } x < x_{i+1/2} \\ \theta(k_{i+1}, u_{i+1}^n) & \text{if } x > x_{i+1/2} \end{array} \right. , & \end{array} \right. \tag{48}$$

where $\hat{k} = (k_i + k_{i+1})/2$ and $\hat{u} = (u_i^n + u_{i+1}^n)/2$. As the Godunov scheme, the flux (which is represented by v) is continuous through the interface $\{x_{i+1/2}\} \times [t^n \times t^{n+1}[$ (this property is obtained by the good choice of θ). If $v_{i+1/2}^n\big((x - x_{i+1/2})/(t - t^n); k_i, k_{i+1}, u_i^n, u_{i+1}^n\big)$ is the exact solution to Riemann problem (48), as the function k is positive, the numerical flux of the VFRoe-ncv scheme is:

$$\begin{array}{rcl} \varphi_{i+1/2}^n & = & v_{i+1/2}^n(0; k_i, k_{i+1}, u_i^n, u_{i+1}^n) \\ & = & \left\{ \begin{array}{ll} \theta(\kappa_i, u_i^n) & \text{if } g'(\hat{u}) > 0 \\ \theta(k_{i+1}, u_{i+1}^n) & \text{if } g'(\hat{u}) < 0, \end{array} \right. \end{array} \tag{49}$$

We can remark that the VFRoe-ncv scheme is reduced to the well-known upwind scheme for problem (48).

Finally, as function g is not genuinely nonlinear, the function g' can be equal to zero on an interval included in $[0,1]$. Then, if $g'(\hat{u}) = 0$, problem (48) is not ill-posed, we take for the numerical flux

$$\varphi_{i+1/2}^n = (k_i g(u_i^n) + k_{i+1} g(u_{i+1}^n))/2. \tag{50}$$

5.3 The God/VFRoe-ncv scheme

We will remark in section 8, that the resolution of the Riemann problem at the interface $\{x_{1/2}\} \times [t^n, t^{n+1}[$ where k is discontinuous, is long and difficult, then we introduce the God/VFRoe scheme.

For $i < 0$ and $i > 0$, the numerical flux is the Godunov flux (defined in subsection 5.1 with (47)):

$$\begin{array}{rcl} \varphi_{i+1/2}^n & = & k_i g(u_{i+1/2}^n(0^-; k_i, k_{i+1}, u_i^n, u_{i+1}^n) \\ & = & k_{i+1} g(u_{i+1/2}^n(0^+; k_i, k_{i+1}, u_i^n, u_{i+1}^n), \end{array} \tag{51}$$

and for $i = 0$, the numerical flux is the VFRoe-ncv flux (defined in subsection 5.2 with (49) and (50)):

$$\begin{array}{rcll} \varphi_{i+1/2}^n & = & v_{i+1/2}^n(0; k_i, k_{i+1}, u_i^n, u_{i+1}^n) & \text{if } g'(\hat{u}) \neq 0, \\ \varphi_{i+1/2}^n & = & (k_i g(u_i^n) + k_{i+1} g(u_{i+1}^n))/2 & \text{if } g'(\hat{u}) = 0. \end{array} \tag{52}$$

6 Numerical results for nor convex neither concave flux function

In this section, numerical results with g nor concave neither convex are presented. The graph of g is the following :

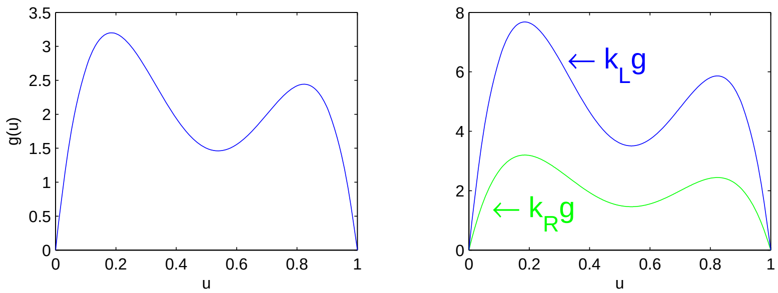

Figure 1: Graph of g and $k_L g$, $k_R g$

For numerical tests, g is given by $g(u) = -23.57u^4 + 48.33u^3 - 32.45u^2 + 7.69x$.

In the two following tests, the Riemann problem is numerically solved. The length of the domain is 10m. The mesh is composed of 100 cells and the CFL condition is set to 0.05. The variable u is plotted, in order to appreciate the behaviour of the Godunov scheme through the interface $\{x/t = 0\}$.
The initial conditions of the first Riemann problem are $k_L = 1.5$, $k_R = 1$, $u_L = 0.53$ and $u_R = 0.4$. The results of Fig. 2 are plotted at $t = 4s$. The analytic solution to this Riemann problem is given in section 8. The numerical approximations provided by the three schemes are similar. We can observe that the three schemes present only one point in the shock between u_L and $u(t = 4s, 0^-)$, moreover this point is in the interval given by $[u_L, u(t = 4s, 0^-)]$.
The initial conditions of the second Riemann problem are $k_L = 1.5$, $k_R = 1$, $u_L = 0.53$ and $u_R = 0.9$. The results of Fig. 3 are plotted at $t = 1s$. The analytic solution to this Riemann problem is given in section 8. The numerical approximation provided by the three schemes are similar and we observe the same behaviour than for the first Riemann problem presented.

7 Numerical results for a piecewise linear flux function

In this section, the function g is defined as follows:

$$g(u) = \left\{ \begin{array}{ccccccc} 4u & \text{if} & 0 & \leq & u & \leq & 1/4, \\ 1 & \text{if} & 1/4 & \leq & u & \leq & 3/4, \\ -4u+4 & \text{if} & 3/4 & \leq & u & \leq & 1, \end{array} \right. \tag{53}$$

We have already remark that problem (1) can be considered as the following resonant system:

$$\partial_t u + \partial_x \big(k(x) g(u)\big) = 0, \quad \partial_t k = 0. \tag{54}$$

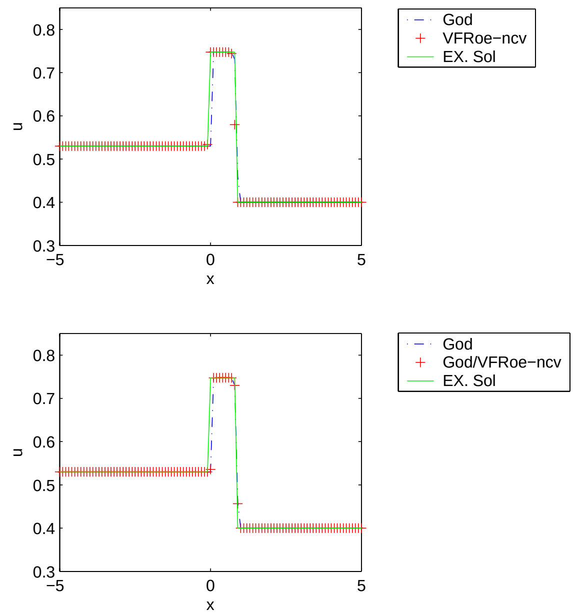

Figure 2: $k_L = 1.5$, $k_R = 1$, $u_L = 0.538$, $u_R = 0.4$, 100 cells.

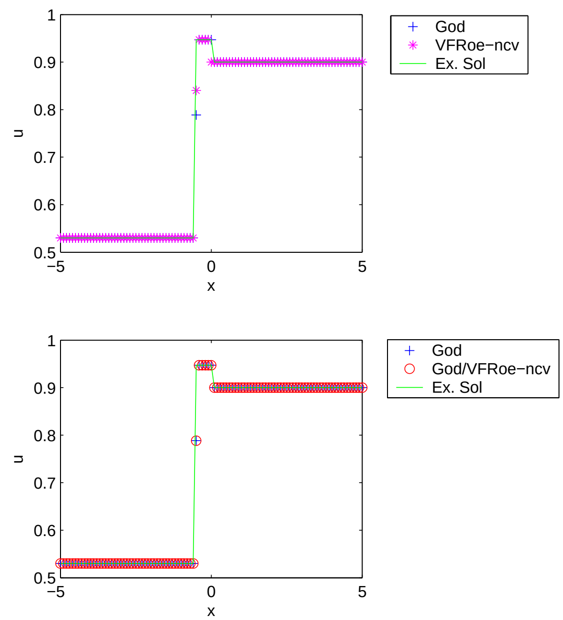

Figure 3: $k_L = 1.5$, $k_R = 1$, $u_L = 0.53$, $u_R = 0.9$, 100 cells.

We notice that the system is resonant for $u \in [1/4, 3/4]$. We will show that the numerical methods are stable in spite of the resonance of the problem.
In the following test, the Riemann problem is numericaly solved. The length of

the domain is 10m. The mesh is composed of 100 cells and the CFL condition is set to 0.12. The solution$u$ is presented in order to appreciate the behaviour of the Godunov and the VFRoe-ncv scheme through the interface $x/t = 0$.
The initial conditions of the two Riemann problems are $k_L = 1.5$, $k_R = 1$, $u_L = 3/8$ and $u_R = 5/8$. We remark that u_L, $u_R \in [1/4, 3/4]$. The results of Fig. 4 are plotted at $t = 2s$. The analytic solution to this Riemann problem is given in section 8. The numerical approximations provided by the Godunov scheme and the VFRoe scheme are similar. We may notice that the VFRoe-scheme introduce an error in the shock between u_L and $u(t = 2, x = 0^-)$. This error is due to the fact that $g'(u_L) = 0$ and $g'(u(t = 2, x = 0^-)) \neq 0$ and isn't due to the discontinuity of function k. Moreover, the behaviour of the scheme God/VFRoe, described in section 5.3, is similar that the behaviour of the Godunov scheme. This error is corrected. Then, even if g is constant on an interval included in $[0, 1]$, the behaviour of the schemes are similar (see seconf picture in Figure 4).

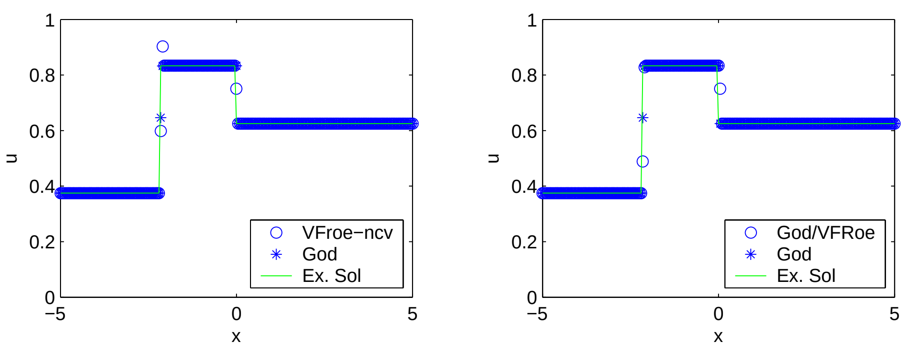

Figure 4: $k_L = 1.5$, $k_R = 1$, $u_L = 3/8$, $u_R = 5/8$, 50 cells

We study now the ability of the schemes to converge towards the entropy solution. On the one hand, with Theorem 3.4 and as the Godunov scheme is monotone and satisfies hypothesis (H7), we know that the approximated solution given by this scheme converges to the entropy solution. But we don't know the order of this scheme. On the other hand, we don't know if the two others schemes are monotone, then Theorem 3.4 can't be use.
The computation of this test are based on the Riemann problem exposed just above. Some measurements of the numerical error provide that the methods tends to zero as Δx tends to zero. The L^1 discrete norm defined as follows: $\Delta x \sum_{i=1..N} |u_i^n - u^{ex}(x_i)|$ is used. But, numerical tests provided by all schemes presented are same behaviour. Several meshes are considering: involving 50, 100, 500, 1000. The axes of Fig. 5 have a logarithmic-scale. We observe a first order convergence for all schemes presented.

Remark 5 We can observe the same results for g presented in section 6.

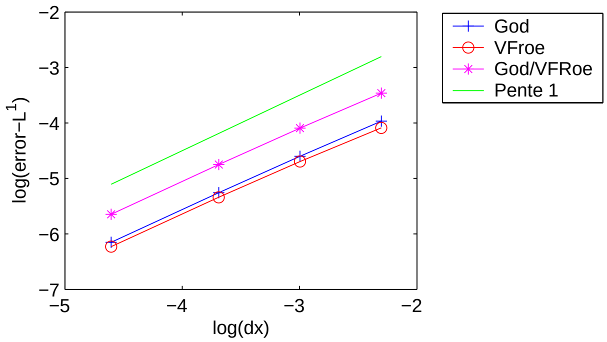

Figure 5: Error estimate in norm L^1

8 The Riemann problem

In this section, the exact solution to the Riemann problem is presented :

$$\begin{cases} \partial_t u + \partial_x\big(k(x)g(u)\big) = 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R}_+ \\ u(t=0,x) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x < 0 \end{cases},\ k(x) = \begin{cases} k_L & \text{if } x < 0 \\ k_R & \text{if } x < 0 \end{cases}, \end{cases}$$

where k_L, $k_R \in \mathbb{R}_+^*$ and u_L, $u_R \in [0,1]$. We note that in [Die95], a general approach to find the solution of Riemann problem is given for fluxes with need not be convex or conave. Solution to the Riemann problem given in section 8.2 agrees with the solution given in [Die95].

8.1 Local entropy condition of the entropy solution

In order to know if a function u is the unique entropy solution of Riemann problem (55), we have to verify that the function u satisfies entropy inequalities (3). These conditions are difficult to satisfy. We can establish equivalent local conditions. In the following, we assume that if $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is an entropy solution, then u admits some traces along the line $\{x = 0\}$ (see [SV03, Bac04]). Let us define $u^- = u(t, x = 0^-)$ and $u^+ = u(t, x = 0^+)$. We can remark that u^- and u^+ are constant. Moreover, in the proof of uniqueness (see [SV03, Bac04]) some properties satisfied by the function u are established :

1. $\forall \kappa \in [0,1]$, $I_u(\kappa) \geq 0$ with

$$I_u(\kappa) = k_L \Phi(u^-, \kappa) - k_R \Phi(u^+, \kappa) + |k_L - k_R| g(\kappa),$$

2. The Rankine-Hugoniot relation

$$k_L g(u^-) = k_R g(u^+). \tag{55}$$

If a function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}; [0,1])$ satisfies these two conditions and if u is a weak solution to problem (1) (with f equal to zero):

$$\int_{R_+} \int_R u(t,x)\partial_t \varphi(t,x) + k(x) g(u(t,x))\partial_x \varphi(t,x)\, dt\, dx \; + \int_R u_0(x)\varphi(0,x)\, dx = 0,$$

then the function u is the unique entropy solution to this problem. Now, we use these two conditions to solve the Riemann problem (55). To describe the solution, we assume for instance $k_L > k_R$.

Let $u_L,\ u_R \in [0,1]$. Let $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}; [0,1])$ be the entropy solution to the Riemann problem (55). Then u satisfies:

<u>For $t \geq 0,\ x < 0$:</u>

- u is the unique entropy solution to:

$$\begin{cases} \partial_t u + \partial_x\big(k_L g(u)\big) = 0 & t \in \mathbb{R}_+,\ x \in \mathbb{R}^*_-, \\ u(t=0,x) = u_L & x \in \mathbb{R}^*_- \\ u(t, x = 0^-) = u^- & t \in \mathbb{R}_+ \end{cases} \tag{56}$$

- If u contains a rarefaction wave, $g'(u(t,x))$ must be negative for $t \in \mathbb{R}_+,\ x \in \mathbb{R}^*_-$.
- If u contains a shock wave, the speed of the shock must be negative.

<u>For $t \geq 0,\ x > 0$:</u>

- u is the unique entropy solution to:

$$\begin{cases} \partial_t u + \partial_x\big(k_R g(u)\big) = 0 & t \in \mathbb{R}_+,\ x \in \mathbb{R}^*_+, \\ u(t=0,x) = u_R & x \in \mathbb{R}^*_+ \\ u(t, x = 0^+) = u^+ & t \in \mathbb{R}_+ \end{cases} \tag{57}$$

- If u contains a rarefaction wave, $g'(u(t,x))$ must be positive for $t \in \mathbb{R}_+,\ x \in \mathbb{R}^*_+$.
- If u contains a shock wave, the speed of the shock must be positive.

<u>For $t \geq 0,\ x = 0$:</u>

- $k_L g(u^-) = k_R g(u^+)$,
- if $u^- \leq u^+$, we only need to verify: $\forall \kappa \in [u^-, u^+]$

$$\begin{aligned} I_u(\kappa) &= -k_L g(u^-) + k_L g(\kappa) - k_R g(u^+) + k_R g(\kappa) + k_L g(\kappa) - k_R g(\kappa) \\ &= 2k_L(g(\kappa) - g(u^-)) \geq 0, \end{aligned}$$

using the Rankine-Hugoniot relation (55).
- if $u^- > u^+$, we only need to verify: $\forall \kappa \in [u^+, u^-]$

$$\begin{aligned} I_u(\kappa) &= k_L g(u^-) - k_L g(\kappa) + k_R g(u^+) - k_R g(\kappa) + k_L g(\kappa) - k_R g(\kappa) \\ &= 2k_R(g(u^+) - g(\kappa)) \geq 0, \end{aligned}$$

using the Rankine-Hugoniot relation (55).

8.2 Solution to the Riemann problem with g nor concave neither convex

In this section, the entropy solution to Riemann problem (55) is described, with g nor concave neither convex. The function g admits two local maximums in α and in γ and one local minimum in β with $\alpha \leq \beta \leq \gamma$ such that $g(\alpha) > g(\gamma) > g(\beta)$. A graph of g is represented in Fig. 8.2.

When the function k is equal to k_0, the construction of the solution to the Riemann problem is necessary. Let u_l and u_r two different states in $[0,1]$. We link u_l and u_r by a shock wave and/or a rarefaction wave. We don't describe all possible situation, but we refer to [Ser96] for more details.

Then, the construction of the solution to Riemann problem (55) is reduced to the determination of u^- and u^+. We only focus on the case $k_L g(\beta) > k_R g(\alpha)$ (for others case, the solution may be constructed by the same way). In fact, with this assumption, the couple of root of $k_L g(u^-) = k_R g(u^+)$ are reduced to two possibilities in several cases:

- <u>if $u_L \leq \alpha$:</u>
  - <u>if $u_R \leq \alpha$ and $k_L g(u_L) \leq k_R g(\alpha)$:</u>
    - $u^- = u_L$,
    - u^+ is the smallest root of $k_R g(u^+) = k_L g(u_L)$ and u^+ and u_R are linked by a shock wave if $k_L g(u_L) > k_R g(u_R)$ or by a rarefaction wave if $k_L g(u_L) < k_R g(u_R)$.
  - <u>if $u_R \leq \alpha$ and $k_L g(u_L) > k_R g(\alpha)$:</u>
    - u^- is the greatest root of $k_L g(u^-) = k_R g(\alpha)$ and u_L and u^- are linked by a shock wave,
    - $u^+ = \alpha$ and u^+ and u_R are linked by a rarefaction wave.
  - <u>if $\alpha < u_R \leq \beta$ and $k_L g(u_L) < k_R g(u_R)$:</u>
    - $u^- = u_L$,
    - u^+ is the smallest root of $k_R g(u^+) = k_L g(u_L)$ and u^+ and u_R are linked by a shock wave.
  - <u>if $\alpha < u_R \leq \beta$, $k_L g(u_L) > k_R g(u_R)$ and $g(u_R) < g(\gamma)$ and $k_L g(u_L) > k_R g(\gamma)$:</u>
    - u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$ and u^- and u_L are linked by a shock wave,
    - $u^+ = \gamma$ and u^+ and u_R are linked by a shock wave.
  - <u>if $\alpha < u_R \leq \beta$, $k_L g(u_L) > k_R g(u_R)$ and $g(u_R) < g(\gamma)$ and $k_L g(u_L) \leq k_R g(\gamma)$:</u>
    - $u^- = u_L$,
    - u^+ is the root of $k_L g(u_L) = k_R g(u^+)$ included in $[\alpha, \beta]$ and u^+ and u_R are linked by a shock wave.
  - <u>if $\alpha < u_R \leq \beta$, $k_L g(u_L) > k_R g(u_R)$ and $g(u_R) \geq g(\gamma)$:</u>

    * u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$ and u^- and u_L are linked by a shock wave,
    * $u^+ = u_R$.
  - <u>if $\beta < u_R \leq \gamma$ and $k_L g(u_L) < k_R g(\beta)$</u>:
    * $u^- = u_L$,
    * u^+ is the smallest root of $k_R g(u^+) = k_L g(u_L)$ and u^+ and u_R are linked by a shock wave, then a rarefaction wave.
  - <u>if $\beta < u_R \leq \gamma$ and $k_L g(u_L) = k_R g(\beta)$</u>:
    * $u^- = u_L$,
    * $u^+ = \beta$ and u^+ and u_R are linked by a rarefaction wave.
  - <u>if $\beta < u_R \leq \gamma$ and $k_L g(u_L) > k_R g(\beta)$ and $k_L g(u_L) < k_R g(u_R)$</u>:
    * $u^- = u_L$,
    * u^+ is the root of $k_L g(u_L) = k_R g(u^+)$ in the interval $[\alpha, \beta]$, and u^+ and u_R are linked by a shock wave.
  - <u>if $\beta < u_R \leq \gamma$ and $k_L g(u_L) > k_R g(\beta)$ and $k_L g(u_L) = k_R g(u_R)$</u>:
    * $u^- = u_L$,
    * $u^+ = u_R$.
  - <u>if $\beta < u_R \leq \gamma$ and $k_L g(u_L) > k_R g(\beta)$ and $k_L g(u_L) > k_R g(u_R)$ and $k_L g(u_L) \leq k_R g(\gamma)$</u>:
    * $u^- = u_L$
    * u^+ is the root of $k_L g(u_L) = k_R g(u^+)$ in $[\beta, \gamma]$, and u_+ and u_R are linked by a rarefaction wave.
  - <u>if $\beta < u_R \leq \gamma$ and $k_L g(u_L) > k_R g(\beta)$ and $k_L g(u_L) > k_R g(u_R)$ and $k_L g(u_L) > k_R g(\gamma)$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$, and u_L et u^- are linked by a shock wave.
    * $u^+ = \gamma$, and u_+ and u_R are linked by a rarefaction wave.
  - <u>$u_R > \gamma$ and $k_L g(u_L) \leq k_R g(\beta)$ and $k_L g(u_L) \leq g(u_R)$</u>:
    * $u^- = u_L$,
    * u^+ is the smallest root of $k_L g(u_L) = k_R g(u^+)$ and u^+ and u_R are linked by a shock.
  - <u>$u_R > \gamma$ and $k_L g(u_L) \leq k_R g(\beta)$ and $k_L g(u_L) > g(u_R)$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$, and u^- and u_L are linked by a shock.
    * $u^+ = u_R$.
  - <u>$u_R > \gamma$ and $k_L g(u_L) > k_R g(\beta)$ and $k_L g(u_L) \leq k_R g(u_R)$</u>:
    * $u^- = u_L$,
    * u^+ is the root of $k_L g(u_L) = k_R g(u^+)$ included in $[\beta, \gamma]$, and u^+ and u_R are linked by a shock wave.

- $u_R > \gamma$ and $k_L g(u_L) > k_R g(\beta)$ and $k_L g(u_L) > k_R g(u_R)$:
    - u^- is the greatest root of $k_R g(u_R) = k_L g(u^-)$ and u_L and u^- are linked by a shock wave,
    - $u^+ = u_R$.

- if $\alpha < u_L \leq \gamma$:
  - if $u_R \leq \alpha$:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(\alpha)$, and u_L and u^- are linked by a rarefaction wave and then a shock wave,
    - $u^+ = \alpha$ and u^+ and u_R are linked by a rarefaction wave.
  - if $\alpha < u_R \leq \beta$ and $k_R g(u_R) \geq k_R g(\gamma)$:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$, and u_L and u^- are linked by a rarefaction wave and then a shock wave,
    - $u^+ = u_R$.
  - if $\alpha < u_R \leq \beta$ and $k_R g(u_R) < k_R g(\gamma)$:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$, and u_L and u^- are linked by a rarefaction wave and then a shock wave,
    - $u^+ = \gamma$, and u^+ and u_R are linked by a shock wave.
  - if $\beta < u_R \leq \gamma$:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$, and u_L and u^- are linked by a rarefaction wave and then a shock wave,
    - $u^+ = \gamma$, and u^+ and u_R are linked by a shock wave.
  - if $\gamma < u_R$:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$, and u_L and u^- are linked by a shock wave,
    - $u^+ = u_R$.

- if $\gamma < u_L$:
  - if $u_R \leq \alpha$:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(\alpha)$, and u_L and u^- are linked by a shock wave if $k_L g(u_L) < k_R g(u_R)$ or by a rarefaction wave if $k_L g(u_L) > k_R g(u_R)$,
    - $u^+ = \alpha$ and u^+ and u_R are linked by a rarefaction wave.
  - if $\alpha < u_R \leq \beta$ and $k_R g(u_R) \geq k_R g(\gamma)$:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$, and u_L and u^- are linked by a shock wave if $k_L g(u_L) < k_R g(u_R)$ or by a rarefaction wave if $k_L g(u_L) > k_R g(u_R)$,
    - $u^+ = u_R$.
  - if $\alpha < u_R \leq \beta$ and $k_R g(u_R) < k_R g(\gamma)$:

* u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$, and u_L and u^- are linked by a shock wave if $k_L g(u_L) < k_R g(\gamma)$ or by a rarefaction wave if $k_L g(u_L) > k_R g(\gamma)$,
* $u^+ = \gamma$, and u^+ and u_R are linked by a shock wave.

– <u>if $\beta < u_R \leq \gamma$:</u>
  * u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$, and u_L and u^- are linked by a shock wave if $k_L g(u_L) < k_R g(\gamma)$ or by a rarefaction wave if $k_L g(u_L) > k_R g(\gamma)$,
  * $u^+ = \gamma$, and u^+ and u_R are linked by a shock wave.

– <u>if $u_R > \gamma$:</u>
  * u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$, and u_L and u^- are linked by a shock wave if $k_L g(u_L) < k_R g(u_R)$ or by a rarefaction wave if $k_L g(u_L) > k_R g(u_R)$,
  * $u^+ = u_R$.

8.3 Explicit form of the solution to the Riemann problem with g piecewise linear

In this section, we describe the entropy solution of Riemann problem (55), with g defined as in section 7 (see Eq. (53)).

We first present the construction when the function k is constant equal to k_0. Let u_l and u_r be two different states in $[0, 1]$. We link u_l and u_r by a shock wave in all case because g is piecewise linear:

$$u(t,x) = \begin{cases} u_l & \text{if} \quad x/t < k_0 \dfrac{g(u_l) - g(u_r)}{u_l - u_r} \\ \\ u_r & \text{if} \quad x/t > k_0 \dfrac{g(u_l) - g(u_r)}{u_l - u_r} \end{cases} \tag{58}$$

The construction of the solution to the Riemann problem is reduced to the determination of u^- and u^+. We only focus on the case $k_L > k_R$ (if $k_L < k_R$, the solution may be constructed by the same way).

- <u>if $u_L < 1/4$</u>
  – <u>if $u_R \leq 1/4$ and $k_L g(u_L) \leq k_R g(1/4)$:</u>
    * $u^- = u_L$
    * u^+ is the smallest root of $k_L g(u_L) = k_R g(u^+)$, and u^+ and u_R are linked by a shock wave (defined by (58) with $u_l = u^+$, $u_r = u_R$ and $k_0 = k_R$).

  – <u>if $u_R < 1/4$ and $k_L g(u_L) > k_R g(1/4)$:</u>
    * u^- is the greatest root of $k_L g(u^-) = k_R g(3/4)$ and u^- and u_L are linked by a shock wave (defined by (58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),

    * $u^+ = 1/4$ and u^+ and u_R are linked by a shock wave (defined by (58) with $u_l = u^+$, $u_r = u_R$ and $k_0 = k_R$).
  - <u>if $u_R = 1/4$ and $k_L g(u_L) > k_R g(1/4)$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(3/4)$ and u^- and u_L are linked by a shock wave (defined by (58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    * $u^+ = u_R$.
  - <u>$1/4 < u_R \leq 3/4$ and $k_L g(u_L) < k_R g(1/4)$</u>:
    * $u^- = u_L$,
    * u^+ is the smallest root of $k_L g(u_L) = k_R g(u^+)$ and u^+ and u_R are linked by a shock wave (defined by (58) with $u_l = u^+$, $u_r = u_R$ and $k_0 = k_R$).
  - <u>$1/4 < u_R \leq 3/4$ and $k_L g(u_L) = k_R g(1/4)$</u>:
    * $u^- = u_L$,
    * $u^+ = u_R$.
  - <u>$1/4 < u_R \leq 3/4$ and $k_L g(u_L) > k_R g(1/4)$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(3/4)$ and u^- and u_L are linked by a shock wave (defined by (58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    * $u^+ = u_R$.
  - <u>$u_R > 3/4$ and $k_L g(u_L) < k_R g(u_R)$</u>:
    * $u^- = u_L$,
    * u^+ is the smallest root of $k_L g(u_L) = k_R g(u^+)$ and u^+ and u_R are linked by a shock wave (defined by (58) with $u_l = u^+$, $u_r = u_R$ and $k_0 = k_R$).
  - <u>$u_R > 3/4$ and $k_L g(u_L) = k_R g(u_R)$</u>:
    * $u^- = u_L$,
    * $u^+ = u_R$.
  - <u>$u_R > 3/4$ and $k_L g(u_L) > k_R g(u_R)$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$ and u^- and u_L are linked by a shock wave (defined by (58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    * $u^+ = u_R$.

- <u>$1/4 \leq u_L \leq 3/4$</u>:
  - <u>$u_R < 1/4$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(1/4)$ and u^- and u_L are linked by a shock wave (defined by (58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    * $u^+ = 1/4$ and u^+ and u_R are linked by shock wave.

- <u>$1/4 \leq u_R$</u>:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$ and u^- and u_L are linked by a shock wave (defined by (58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    - $u^+ = u_R$.

- <u>$u_L > 3/4$</u>:
  - <u>$u_R < 1/4$ and $k_L g(u_L) > k_R g(u_R)$</u>:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(1/4)$ and u^- and u_L are linked by a shock wave (defined by (58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    - $u^+ = 1/4$, and u^+ and u_R are linked by a shock wave.
  - <u>$u_R < 1/4$ and $k_L g(u_L) \leq k_R g(u_R)$</u>:
    - u^- is the smallest root of $k_L g(u^-) = k_R g(u_R)$ and u^- and u_L are linked by a shock wave (defined by (58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    - $u^+ = 1/4$, u^+ and u_R are linked by a shock wave.
  - <u>$u_R < 1/4$ and $k_L g(u_L) < k_R g(u_R)$</u>:
    - u^- is the smallest root of $k_L g(u^-) = k_R g(u_R)$ and u^- and u_L are linked by a shock wave (defined by (58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    - $u^+ = u_R$.
  - <u>$1/4 \leq u_R$</u>:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$ and u^- and u_L are linked by a shock wave (defined by (58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    - $u^+ = u_R$.

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Bibliography

[AV03] Adimurthi and G.D. Veerappa Gowda, *Conservation law with discontinuous flux*, J. of Mathematics of Kyoto Univ., **43**, (2003), no. 1, 27-70.

[AJV04] Adimurthi, J. Jaffre and G.D. Veerappa Gowda, *Godunov-type methods for conservation laws with a flux function discontinuous in space time*, SIAM J; Numer Anal., **42**, (2004), no. 1, 179-208.

[Bac04] F. Bachmann, *Analysis of a scalar conservation law with a flux function with discontinuous coefficients*, Advances in Differential Equations (nov-déc 2004), no. 11-12, 1317–1338.

[BV05] F. Bachmann and J. Vovelle *Existence and uniqueness of entropy solution of scalar conservation law with a flux function involving discontinuous coefficients*, to appear in Communications in Partial Differential Equations.

[BGH00] T. Buffard, T. Gallouët and J.M. Hérard, *A sequel to a rough Godunov scheme. Application to real gas flows*, Computers and Fluids, **29-7**, (2000), 813-847.

[BKRT04] R. Bürger, K.H. Karlsen, N.H. Risebro and J.D. Towers, *Well-possedness in BV_t and convergence of a difference scheme for continuous sedimentation in ideal clarifier-tickener units*, Numer. Math., **97**, (2004), no. 1, 25-65.

[Die95] S. Diehl, *A conservation law with point source and discontinuous flux function modelling continuous sedimentation*, SIAM J. Appl. Math., **56**, (1996), no. 2, 388-419.

[DiP85] R.J. DiPerna, *Measure-valued solutions to conservation laws*, Arch. Rational Mech. Anal., **88**, (1985), no. 3, 223–270.

[EGH00] R. Eymard, T. Gallouët, and R. Herbin, *Finite volume methods*, Handbook of numerical analysis, Vol. VII, North-Holland, Amsterdam, 2000, pp. 713–1020.

[GHS02] T. Gallouët, J.M. Hérard and N. Seguin, *Some recent Finite Volume schemes to compute Euler equations using real gas EOS*, Int. J. Num. Meth. Fluids, **39-12**, (2002), 1073-1138.

[God59] S. K. Godunov, *A difference method for numerical calculation of discontinuous solutions of the equations hydrodynamics*, Mat. Sb. (N.S.) **47 (89)** (1959), 271-306.

[KRT02a] K.H. Karlsen, N.H. Risebro and J.D Towers. *Upwind difference approximations for degenerate parabolic convection-diffusion equations with discontinuous coefficient* IMA J. Numer. Anal., **22**, (2002), no. 4,623-664.

[KRT02b] K.H. Karlsen, N.H. Risebro and J.D Towers. *On a nonlinear degenerate parabolic transport-diffusion equation with a discontinuous coefficient.* Electron J. Differential Equations, (2002), pages No. 93, 23 pp (electronic).

[KRT03] K.H. Karlsen, N.H. Risebro and J.D Towers. *L1 stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients*, Skr. K. Nor. Vid.Selsk, 1–49, 2003.

[KT04] K.H. Karlsen and J.D. Towers, *Convergence of the Lax-Friedrichs Scheme and Stability for Conservation Laws with a Discontinuous Space-Time Dependent Flux*, Chinese Ann. Math. Ser. B., **25**, (2004), no. 3, 287-318.

[KR01] K.H. Karlsen and N.H. Risebro. *Stability of a resonant system of conservation laws modelling polymer flow with gravitation*, J. Differential Equations, **120**, (2001), no. 2, 344-380.

[KR95] K.H. Karlsen and N.H. Risebro. *Convex conservation laws with discontinuous coefficients. Existence, uniqueness ans asymptotic behavior*, Comm. Partial Differential Equations, **20**, (1995), 11-12, 1959-1990.

[LTW95a] L.W. Lin, B.J. Temple, J.H. Wang, *A Comparison of convergence rates for Godunov's methods and Glimm's methods in resonant nonlinear systems of conservations laws*, SIAM J. Numer. Anal., **32**, (1995), 824-840.

[LTW95b] L.W. Lin, B.J. Temple, J.H. Wang, *Suppression of oscillations in Godunov's method for a resonant non-strictly hyperbolic system*, SIAM J. Numer. Anal., **32**, (1995), 841-864.

[SV03] N. Seguin and J. Vovelle, *Analysis and approximation of a scalar conservation law with a flux function with discontinuous coefficients*, Math. Models Methods Appl. Sci., **13**, (2003), no. 2, 221–257.

[Ser96] D. Serre, *Systèms de lois de conservations I*, Diderot Editeur, Arts et Sciences, (1996).

[Tow00] J.D. Towers, *Convergence of a difference scheme for conservation laws with a discontinuous flux*, SIAM J. Numer. Anal., **38**, (2000), no. 2, 681–698 (electronic).

[Tow01] J.D. Towers, *A difference scheme for conservation laws with a discontinuous flux: the nonconvex case*, SIAM J. Numer. Anal., **39**, (2001), no. 4, 1197–1218 (electronic).