A staggered finite volume scheme on general meshes for the Navier-Stokes equations in two space dimensions

Robert Eymard*

Université de Marne-la-Valle, 5, boulevard Descartes, Champs-sur-Marne, F-77454 MARNE LA VALLE CEDEX 2, France

Robert.Eymard@univ-mlv.fr

Raphaèle Herbin*

LATP, Université de Provence, 33 rue Joliot Curie, 13453 Marseille, France

Raphaèle Herbin@cmi.univ-mrs.fr

Abstract

This paper presents a new finite volume scheme for the incompressible steady-state Navier-Stokes equations on a general 2D mesh. The scheme is staggered, i.e. the discrete velocities are not located at the same place as the discrete pressures. We prove the existence and the uniqueness of a discrete solution for a centered scheme under a condition on the data, and the unconditional existence of a discrete solution for an upstream weighting scheme. In both cases (nonlinear centered and upstream weighting schemes), we prove the convergence of a penalized version of the scheme to a weak solution of the problem. Numerical experiments show the efficiency of the schemes on various meshes.

Key words : Navier-Stokes equations, cell-centered finite volumes, unstructured mesh.

1 Introduction

We study the following problems: find an approximation of $u = (u^{(1)}, u^{(2)})^t \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $p \in L^2(\Omega)$, weak solution to the generalized incompressible steady–state Navier-Stokes equations, which write:

$$\begin{aligned} &\eta u - \nu \Delta u + \nabla p + (u \cdot \nabla) u = f \text{ in } \Omega, \\ &\mathrm{div} u = 0 \text{ in } \Omega, \end{aligned} \tag{1}$$

where $\eta \geq 0$, $u^{(1)}$ and $u^{(2)}$ are the two components of the velocity, p denotes the pressure, ν the viscosity of the fluid, under the following assumptions:

$$\Omega \text{ is a polygonal open bounded connected subset of } \mathbb{R}^2, \tag{2}$$

$$\nu \in (0, +\infty), \ \ \eta \in [0, +\infty), \tag{3}$$

$$f = (f^{(1)}, f^{(1)})^t \in (L^2(\Omega))^2, \ \ \text{for } i = 1, 2, \tag{4}$$

and where, by definition, $(u \cdot \nabla)u_i = u^{(1)}\partial_1 u^{(i)} + u^{(2)}\partial_2 u^{(i)}$.The terms $\eta u^{(i)}$ appear when considering an implicit time discretization of the unsteady Navier-Stokes equations (with η as the inverse of the time step). The case $\eta = 0$ yields the usual steady-state equations.

For the simplicity of this presentation, we prescribe a homogeneous Dirichlet boundary condition on the velocity $(u^{(1)}, u^{(2)})$. In all this paper, we denote by $x = (x^{(1)}, x^{(2)})$ any point of Ω and by dx the 2-dimensional Lebesgue measure $dx = dx^{(1)}dx^{(2)}$.

Definition 1.1 (Weak solution) Under hypotheses (2)-(4), let

$$E(\Omega) := \{v = (v^{(1)}, v^{(2)})^t \in (H_0^1(\Omega))^2, \mathrm{div} v = \partial_1 v^{(1)} + \partial_2 v^{(2)} = 0 \text{ a.e.}\}. \tag{5}$$

Then $u = (u^{(1)}, u^{(2)})^t$ is called a weak solution of (1) (see e.g. [24]) if and only if

$$\left\{ \begin{array}{l} u = (u^{(1)}, u^{(2)})^t \in E(\Omega), \\ \eta \displaystyle\int_\Omega u(x)v(x)dx + \nu \int_\Omega \nabla u : \nabla v(x)dx + b(u, u, v) = \\ \displaystyle\int_\Omega f(x) \cdot v(x)dx, \quad \forall v = (v^{(1)}, v^{(2)})^t \in E(\Omega), \end{array} \right. \tag{6}$$

where, by definition, $\nabla u : \nabla v(x) = \displaystyle\sum_{i=1,2} \nabla u^{(i)}(x) \cdot \nabla v^{(i)}(x)$. and where the trilinear form b is defined for all $u, v, w \in (H_0^1(\Omega))^2$ by

$$b(u, v, w) = (u \cdot \nabla v)w = \sum_{k=1,2} \sum_{i=1,2} \int_\Omega u^{(i)}(x)\partial_i v^{(k)}(x)w^{(k)}(x)dx, \tag{7}$$

which classically satisfies, for all $u \in E(\Omega)$,

$$b(u, v, w) = \sum_{k=1,2} \sum_{i=1,2} \int_\Omega \partial_i(u^{(i)}v^{(k)})(x)w^{(k)}(x)dx.$$

Numerical schemes for Navier-Stokes equations (6)-(7) have been extensively studied: see [12, 21, 22, 23, 14, 13] and references therein. As we noted in [4], an advantage of the finite volume schemes is that the unknowns are approximated by piecewise constant functions, and indeed, the classical finite volume scheme on rectangular meshes has been the basis of many industrial applications. However, the use of rectangular grids makes an important limitation to the type of domain which can be gridded and more recently, finite volume schemes for the Navier-Stokes equations on triangular grids have been presented: [15], [6].

In this paper, we propose a method which uses the primitive variables and enforces the divergence condition directly, using quite general meshes such as mixed rectangular-triangular or Voronoï meshes. The method used here is in fact a generalization to the nonlinear case of a staggered finite volume scheme which we introduced in [4] and for which we proved convergence. In this scheme, the discrete unknowns are the discrete velocities located at some point within the discretization cells (or "control volumes" of the mesh (see [4]) whereas the discrete divergence–free condition is imposed at the vertices of the mesh. The additional difficulty which is addressed here, is the discretization of the trilinear form $b(u, u, v)$, defined by (7), in a way which enables us obtain estimates and prove convergence.

This paper is organized as follows. The finite volume scheme is presented in Section 2, using the notations and definitions which were introduced in Section 2 of [4]. We propose two means to discretize the nonlinear form b, namely a centered scheme and an upstream weighting scheme. Although it is wellknown that the upstream scheme is not as precise as the centered scheme, it is however often used when the convection terms are dominant with respect to the grid size, in order to prevent oscillations. For both cases we prove the existence of the discrete velocity; note that this existence is unconditional for the upstream scheme, but that it only holds under a small data (or minimal viscosity) condition for the centered scheme. In the case of the centered scheme, we also prove (under a small data condition) the uniqueness for both the penalized and the non penalized scheme.

We then prove the convergence of the solutions to the centered and the upstream schemes, as the mesh size tends to 0, for penalized versions of the schemes, with the same restriction on the data for the centered scheme. We give some numerical results in Section 3, and finally conclude with some remarks on open problems (Section 4).

2 The finite volume scheme

We now turn to the study of the finite volume scheme for the steady Navier-Stokes equations. Throughout the proofs of existence, uniqueness and convergence of both the centered and upstream schemes, we shall need to compare the values of the velocities at the vertices of the mesh and at the centers of the control volumes because of the discretization of the trilinear form. We thus make use of the following proposition, where a control of the difference between the discrete grid function and its reconstruction on the dual grid is proven, in both the mesh-dependent discrete H^1 norm and the L^2 norms.

PROPOSITION 2.1 **[Control of the reconstruction of velocities on the dual grid]** Under hypothesis (2), Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 of [4] and let $\alpha > 0$ such that $\text{angle}(\mathcal{D}) \geq \alpha$. Let $u \in H_{\mathcal{D}}(\Omega)$, and let $(c_{K,S})_{S\in\mathcal{V},\ K\in\mathcal{M}_S}$ be a family of nonnegative real values such that

$$\sum_{K\in\mathcal{M}_S} c_{K,S} = 1,\ \forall S \in \mathcal{V},$$

and let $\tilde{u} \in L_{\mathcal{D}}(\Omega)$ be defined by

$$\tilde{u}_S = \sum_{K \in \mathcal{M}_S} c_{K,S}\, u_K,\ \forall S \in \mathcal{V}.$$

Then there exists $C_1 > 0$ which only depends on α, such that

$$\sum_{S \in \mathcal{V}} \sum_{K \in \mathcal{M}_S} (\tilde{u}_S - u_K)^2 \le C_1\, |u|_{\mathcal{D}}^2, \tag{8}$$

$$\|u - \tilde{u}\|_{L^2(\Omega)}^2 \le h^2 C_1\, |u|_{\mathcal{D}}^2 \tag{9}$$

and

$$\|\tilde{u}\|_{L^2(\Omega)}^2 \le \mathrm{diam}(\Omega)^2 (C_1 + 1) |u|_{\mathcal{D}}^2 \tag{10}$$

Proof Let $S \in \mathcal{V}$ and $\bar{K} \in \mathcal{M}_S$. Let us write $\mathcal{M}_S = \{K_1, K_2, \ldots, K_m\}$, assuming that, for all $i = 1, \ldots, m-1$, $\mathcal{E}_{K_i} \cap \mathcal{E}_{K_{i+1}} \neq \emptyset$, and $\bar{K} = K_1$. Thanks to the hypothesis on the regularity on $\mathcal{D}$, we can apply inequality (11) of given by Proposition 2.1 of [4], which yields $m \le C_2$ where C_2 only depends on α. We have

$$\tilde{u}_S - u_{\bar{K}} = \sum_{i=1}^{m} c_{K_i,S}\,(u_{K_i} - u_{K_1}) = \sum_{i=2}^{m} c_{K_i,S} \sum_{j=1}^{i-1} (u_{K_{j+1}} - u_{K_j}) = \sum_{j=1}^{m-1} (u_{K_{j+1}} - u_{K_j}) \sum_{i=j+1}^{m} c_{K_i,S}.$$

Since $0 \le \sum_{i=j+1}^{m} c_{K_i,S} \le 1$, for all $j = 1, \ldots, m-1$, we get, using the Cauchy-Schwarz inequality, that

$$(\tilde{u}_S - u_{\bar{K}})^2 \le C_2 \sum_{j=1}^{m-1} (u_{K_{j+1}} - u_{K_j})^2.$$

We introduce the function $\chi(\sigma, S)$ such that $\chi(\sigma, S) = 1$ if $\sigma \in \mathcal{E}_S$ and $\chi(\sigma, S) = 0$ otherwise. Since $(u_K - u_L)^2 \le (u_K - u_\sigma)^2 + (u_L - u_\sigma)^2$ for $K|L = \sigma$, the above inequality yields:

$$(\tilde{u}_S - u_{\bar{K}})^2 \le C_2 \sum_{K \in \mathcal{M}_S} \sum_{\sigma \in \mathcal{E}_K} (u_K - u_\sigma)^2 \chi(\sigma, S).$$

Thus we get

$$\sum_{S \in \mathcal{V}} \sum_{\bar{K} \in \mathcal{M}_S} (\tilde{u}_S - u_{\bar{K}})^2 \le C_2 \sum_{S \in \mathcal{V}} \sum_{\bar{K} \in \mathcal{M}_S} \sum_{K \in \mathcal{M}_S} \sum_{\sigma \in \mathcal{E}_K} (u_K - u_\sigma)^2 \chi(\sigma, S),$$

which yields

$$\sum_{S \in \mathcal{V}} \sum_{\bar{K} \in \mathcal{M}_S} (\tilde{u}_S - u_{\bar{K}})^2 \le C_2 \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (u_K - u_\sigma)^2 \sum_{S \in \mathcal{V}_K} \chi(\sigma, S) \sum_{\bar{K} \in \mathcal{M}_S} 1.$$

Since $\sum_{S \in \mathcal{V}_K} \chi(\sigma, S) = 2$ and since $\sum_{\bar{K} \in \mathcal{M}_S} 1 \le C_2$, we get

$$\sum_{S \in \mathcal{V}} \sum_{K \in \mathcal{M}_S} (\tilde{u}_S - u_K)^2 \le 2{C_2}^2 \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (u_K - u_\sigma)^2.$$

Thanks to the inequality (10) given in Proposition 2.1 of [4], we get the existence of C_1 , which only depends on α, such that

$$\sum_{K\in\mathcal{M}}\sum_{S\in\mathcal{V}_K}(\tilde{u}_S-u_K)^2\le C_1\,|u|^2_{\mathcal{D}}, \tag{11}$$

which proves (8). Since we have

$$\|u-\tilde{u}\|^2_{L^2(\Omega)}=\int_\Omega(u(x)-\tilde{u}(x))^2dx=\sum_{K\in\mathcal{M}}\sum_{S\in\mathcal{V}_K}\mathrm{m}(K\cap S)(u_K-\tilde{u}_S)^2,$$

we get, using $\mathrm{m}(K\cap S)\le \mathrm{diam}(K)^2$ and (11),

$$\|u-\tilde{u}\|^2_{L^2(\Omega)}\le h^2C_1\,|u|^2_{\mathcal{D}}.$$

This concludes the proof of (9). Using the discrete Poincaré inequality (see [8] or (20) in [4]) and $h\le\mathrm{diam}(\Omega)$, we get (10). □

2.1 The centered scheme

Under hypotheses (2)-(4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 of [4]. Let $\lambda\in(0,+\infty)$ be given. The finite volume scheme for the approximation of the solution (1) is defined by the following set of equations: find u such that

$$\begin{aligned}&u\in E_{\mathcal{D}}(\Omega),\\&\eta\int_\Omega u(x)\cdot v(x)dx+\nu[u,v]_{\mathcal{D}}+b_{\mathcal{D}}(u,u,v)=\int_\Omega f(x)\cdot v(x)dx,\ \forall v\in E_{\mathcal{D}}(\Omega),\end{aligned} \tag{12}$$

where, for u,v and $w\in H_{\mathcal{D}}(\Omega)$, we define, in this section, the following centered approximation $b_{\mathcal{D}}$ of the trilinear form b defined by (7):

$$\begin{aligned}b_{\mathcal{D}}(u,v,w)&=\sum_{K\in\mathcal{M}}\sum_{k=1,2}w_K^{(k)}\sum_{S\in\mathcal{V}_K}v_S^{(k)}\sum_{i=1,2}A^{(i)}_{K,S}\,u_K^{(i)}\\v_S^{(k)}&=\frac{1}{\mathrm{m}(S)}\sum_{K\in\mathcal{M}_S}\mathrm{m}(K\cap S)\;v_K^{(k)},\ \forall S\in\mathcal{V},\ k=1,2.\end{aligned} \tag{13}$$

(Recall that

$$\begin{aligned}A^{(1)}_{K,S}&=x^{(2)}_{\sigma_1}-x^{(2)}_{\sigma_2}\\A^{(2)}_{K,S}&=x^{(1)}_{\sigma_2}-x^{(1)}_{\sigma_1},\end{aligned} \tag{14}$$

see [4] for the notations.

Let us first prove that the trilinear form $b_{\mathcal{D}}(u,v,w)$ satisfies some continuity properties in $(H_{\mathcal{D}}(\Omega))^3$.

LEMMA 2.2 **[Continuity of the trilinear form in discrete H^1 space]** Under Hypothesis (2), let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.1 of [4], let $\alpha>0$ be such that $\mathrm{angle}(\mathcal{D})\ge\alpha$, let $H_{\mathcal{D}}(\Omega)$ be the space of piecewise constant functions defined in Definition 3.1 of [4], and let $b_{\mathcal{D}}$ be the trilinear form defined by (13).

Then there exists $C_3>0$, only depending on α, such that:

$$|b_{\mathcal{D}}(u,v,w)|\le C_3\,|u|_{\mathcal{D}}|v|_{\mathcal{D}}|w|_{\mathcal{D}}. \tag{15}$$

Proof Let $u, v, w \in H_{\mathcal{D}}(\Omega)$. Since the equalities $\displaystyle\sum_{i=1,2}\sum_{S\in\mathcal{V}_K} A^{(i)}_{K,S} = 0$ hold for any $K \in \mathcal{M}$, for $i = 1, 2$, we have

$$b_{\mathcal{D}}(u,v,w) = \sum_{k=1,2}\sum_{K\in\mathcal{M}}\sum_{S\in\mathcal{V}_K} w^{(k)}_K (v^{(k)}_S - v^{(k)}_K) \sum_{i=1,2} A^{(i)}_{K,S} u^{(i)}_K .$$

Applying the Cauchy-Schwarz inequality to the above equation and using Proposition 2.1 yields the existence of C_4 depending on α such that

$$b_{\mathcal{D}}(u,v,w)^2 \le C_4 \, |u_{\mathcal{D}}|^2 \sum_{k=1,2}\sum_{K\in\mathcal{M}}\sum_{S\in\mathcal{V}_K} \left(w^{(k)}_K \sum_{i=1,2} A^{(i)}_{K,S} u^{(i)}_K \right)^2 . \tag{16}$$

By Definition 14, we have $|A^{(i)}_{K,S}| \le \mathrm{diam}(K)$, for $i = 1, 2$. Thanks to Proposition 2.1 of [4], we thus get $(A^{(i)}_{K,S})^2 \le C_5 \, \mathrm{m}(K)$ for $i = 1, 2$, where C_5 depends only on α.

Using the Cauchy-Schwarz inequality on the right hand side of (16) and thanks to (12) in Proposition 2.1 of [4], one obtains the following inequality:

$$b_{\mathcal{D}}(u,v,w)^2 \le C_6 \, |u_{\mathcal{D}}|^2 \sum_{k=1,2} \left(\|w^{(k)}\|_{L^4(\Omega)} \sum_{i=1,2} \|u^{(i)}\|_{L^4(\Omega)} \right)^{\frac{1}{2}} ,$$

where C_6 depends only on α. The discrete Sobolev inequalities [7, 8] then yield (15). □

As in the case of the linear problem, we use the following penalized approximation of (12):

$$\begin{aligned}
&(u,p) \in (H_{\mathcal{D}}(\Omega))^2 \times L_{\mathcal{D}}(\Omega),\\
&\nu\left([u,v]_{\mathcal{D}}\right) - \int_\Omega p(x)\mathrm{div}_{\mathcal{D}}(v)(x)dx + b_{\mathcal{D}}(u,u,v) = \int_\Omega f(x)\cdot v(x)dx, \quad \forall v \in (H_{\mathcal{D}}(\Omega))^2,\\
&\mathrm{div}_{\mathcal{D}}(u) = -\lambda\, h\, p,
\end{aligned} \tag{17}$$

Remark 1 System (17) is a set of nonlinear equations with unknowns $(u^{(i)}_K)_{K\in\mathcal{M}}$, $i = 1, 2$ and $(p_S)_{S\in\mathcal{V}}$.

The following proposition gives a sufficient condition for the existence and uniqueness of a solution to the scheme (with or without penalization), under the classical assumption that the data are small, or the viscosity is large enough (see [24] Theorem 1.3 page 167 for the continous case, see also [16]). Note that in the continuous case, the “small data” assumption is only required to prove uniqueness, not existence. Here, however, this assumption is also required for the existence of a discrete solution. Moreover, uniqueness is only proven for ”small enough” solutions.

Proposition 2.3 **[Existence and uniqueness of small discrete solutions in the small data case, with or without a penalization]** Under hypotheses (2)-(4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 of [4] and let $\alpha > 0$ with $\text{angle}(\mathcal{D}) \geq \alpha$. Let C_3 be the real value which only depends on α, given by (15) of Lemma 2.2. Assume that the condition

$$\frac{1}{\nu^2}\left(\sum_{i=1,2}\|f^{(i)}\|_{L^2(\Omega)}\right) < C_7 := \frac{1}{4\text{diam}(\Omega)C_3} \tag{18}$$

is fulfilled. Then there exists one and only one function $u \in (H_{\mathcal{D}}(\Omega))^2$ such that

$$|u|_{\mathcal{D}} \leq C_8 := \frac{1}{2C_3}\left[\nu - \left(\nu^2 - 4\left(\sum_{i=1,2}\|f^{(i)}\|_{L^2(\Omega)}\right)\text{diam}(\Omega)C_3\right)^{1/2}\right], \tag{19}$$

and u is solution to (12) and (13) (no penalization), or u is such that there exists a function p with (u, p) solution to (17) and (13) for a given $\lambda \in (0, +\infty)$. Furthermore, in the latter case, the following inequality holds:

$$\lambda\ h\ \|p\|^2_{L^2(\Omega)} \leq \text{diam}(\Omega)\left(\sum_{i=1,2}\|f^{(i)}\|_{L^2(\Omega)}\right) + C_3\, C_8{}^2. \tag{20}$$

and the function p is unique too.

Proof Let us first handle the nonpenalized case $\lambda = 0$. We consider the mapping

$$u \in (H_{\mathcal{D}}(\Omega))^2 \mapsto \tilde{u} \in (H_{\mathcal{D}}(\Omega))^2$$

such that

$$\begin{aligned} &\tilde{u} \in E_{\mathcal{D}}(\Omega) \\ &\nu[\tilde{u}, v]_{\mathcal{D}} + b_{\mathcal{D}}(u, u, v) = \int_\Omega f(x)\cdot v(x)dx, \quad \forall v \in E_{\mathcal{D}}(\Omega). \end{aligned} \tag{21}$$

Indeed, the existence and uniqueness of $\tilde{u}$ is a straightforward consequence of Proposition 5.2 of [4]. We get, setting $v = \tilde{u}$ in (21), using the discrete Poincaré inequality (see [8] or (20) in [4]) and Lemma 2.2,

$$\nu|\tilde{u}|_{\mathcal{D}} \leq \text{diam}(\Omega)\left(\sum_{i=1,2}\|f^{(i)}\|_{L^2(\Omega)}\right) + C_3\,|u|^2_{\mathcal{D}}. \tag{22}$$

The assumption (18) is equivalent to

$$\nu^2 - 4C_3\,\text{diam}(\Omega)\left(\sum_{i=1,2}\|f^{(i)}\|_{L^2(\Omega)}\right) > 0.$$

Thus the value $C_8 > 0$ defined in (19) is the smallest root of the second degree polynomial

$$P(X) = C_3\,X^2 - \nu X + \text{diam}(\Omega)\left(\sum_{i=1,2}\|f^{(i)}\|_{L^2(\Omega)}\right). \tag{23}$$

Note that C_8 tends to 0 as $\|f\|_{(L^2(\Omega))^2}$ tends to 0, with fixed other parameters. Therefore, we get from (22) that, for $|u|_{\mathcal{D}} \le C_8$, $|\tilde{u}|_{\mathcal{D}} \le C_8$ holds. We can therefore apply Brouwer's fixed point theorem, which yields the existence of $u \in (H_{\mathcal{D}}(\Omega))^2$ which satisfies (12). Let us now show that u is unique. Let us suppose that u and w are two solutions to (12) such that (19) holds. Let $\hat{u}$ denote the difference between the two solutions, that is $\hat{u} = u - w$. Then $\hat{u}$ satisfies:

$$\nu[\hat{u}, v]_{\mathcal{D}} + b_{\mathcal{D}}(u, u, v) - b_{\mathcal{D}}(w, w, v) = 0, \ \forall v \in (H_{\mathcal{D}}(\Omega))^2. \tag{24}$$

We then set $v = \hat{u}$ in the above equation. Thanks to the trilinearity property of $b_{\mathcal{D}}$, we have

$$b_{\mathcal{D}}(u, u, \hat{u}) - b_{\mathcal{D}}(w, w, \hat{u}) = b_{\mathcal{D}}(\hat{u}, u, \hat{u}) + b_{\mathcal{D}}(w, \hat{u}, \hat{u}).$$

Therefore we have:

$$\nu \left(|\hat{u}|^2_{\mathcal{D}}\right) \le -(b_{\mathcal{D}}(\hat{u}, u, \hat{u}) + b_{\mathcal{D}}(w, \hat{u}, \hat{u})),$$

which leads, using (15) and (19) for u and w, to

$$\nu|\hat{u}|^2_{\mathcal{D}} \le 2C_8\, C_3\, |\hat{u}|^2_{\mathcal{D}}.$$

Let us assume $|\hat{u}|_{\mathcal{D}} \ne 0$. On the one hand, the previous inequality produces

$$\nu \le 2C_8\, C_3\,.$$

But on the other hand, we noticed that C_8 is the smallest root of the polynomial P defined by (23). It is therefore strictly smaller than the average value of both roots, that is $\nu/(2C_3)$. This contradiction with the above inequality implies that $\hat{u} = 0$, which proves the uniqueness of solutions satisfying (19).

Let us now handle the case of the penalized scheme, i.e. $\lambda \in (0, +\infty)$. We exactly follow the same steps as above, replacing the definition (21) of $\tilde{u}$ by

$$\begin{aligned}
&(\tilde{u}, \tilde{p}) \in (H_{\mathcal{D}}(\Omega))^2 \times L_{\mathcal{D}}(\Omega),\\
&\nu[\tilde{u}, v]_{\mathcal{D}} - \int_\Omega \tilde{p}(x)\mathrm{div}_{\mathcal{D}}(v)(x)dx + b_{\mathcal{D}}(u, u, v) = \int_\Omega f(x)\cdot v(x)dx, \quad \forall v \in (H_{\mathcal{D}}(\Omega))^2,\\
&\mathrm{div}_{\mathcal{D}}(\tilde{u}) = -\lambda\ h\ \tilde{p}.
\end{aligned} \tag{25}$$

Indeed, the existence and uniqueness of $(\tilde{u}, \tilde{p})$ is then a straightforward consequence of Corollary 5.1 of [4]. We then get (22), setting $v = \tilde{u}$ in (25). The proof of existence and uniqueness of the solution to (17) and (13) is then similar to that of the case $\lambda = 0$, using

$$\begin{aligned}
&\nu\left([\hat{u}, v]_{\mathcal{D}}\right) + \frac{1}{\lambda\ h}\int_\Omega \mathrm{div}_{\mathcal{D}}(\hat{u})(x)\mathrm{div}_{\mathcal{D}}(v)(x)dx + b_{\mathcal{D}}(u, u, v) - b_{\mathcal{D}}(w, w, v) = 0,\\
&\forall v \in (H_{\mathcal{D}}(\Omega))^2,
\end{aligned}$$

instead of (24). Then (20) and the uniqueness of p immediately result from (17).

□

As in the case of the linear problem, we may prove the convergence of the scheme to the continuous solution for the scheme (17), (13). This is stated in the next proposition.

Proposition 2.4 [**Convergence of the centered penalized scheme in the nonlinear case**] Under Hypotheses (2)-(4), let $\alpha > 0$ be given and let $C_7 > 0$ be given by Proposition 2.3. We assume that the property (18) holds. Let $\lambda \in (0, +\infty)$ be given and let $(\mathcal{D}^{(n)})_{n\in\mathbb{N}}$ be a sequence of admissible discretization of Ω in the sense of Definition 2.1 of [4], such that $\lim_{n\to\infty} \text{size}(\mathcal{D}^{(n)}) = 0$ and $\text{angle}(\mathcal{D}^{(n)}) \geq \alpha$, for all $n \in \mathbb{N}$. Let $(u^{(n)}, p^{(n)}) \in (H_{\mathcal{D}^{(n)}}(\Omega))^2 \times L_{\mathcal{D}^{(n)}}(\Omega)$ be a solution to (17), (13),(19). Then there exists a subsequence of the sequence $(u^{(n)})_{n\in\mathbb{N}}$ which converges in $L^2(\Omega)^2$ to u, weak solution of the Navier-Stokes problem in the sense of (6). If C_7 is taken small enough, the uniqueness property of the solution entails the convergence of the whole sequence.

Proof We proceed in a similar way to that of the proof of convergence of the velocities in the linear case, see Proposition 7.1 of [4]. Using the same notations, the only additional property which must be proved is that

$$\lim_{n\to\infty} b_{\mathcal{D}^{(n)}}(u^{(n)}, u^{(n)}, \varphi^{(n)}) = b(u, u, \tilde{\varphi}). \tag{26}$$

Now

$$b_{\mathcal{D}}(u^{(n)}, u^{(n)}, \varphi^{(n)}) = \sum_{K\in\mathcal{M}} \sum_{k=1,2} \varphi_K^{(k)} \sum_{S\in\mathcal{V}_K} u_S^{(k)} \sum_{i=1,2} A_{K,S}^{(i)}\, u_K^{(i)}, \tag{27}$$

where $u_S^{(k)}$ is defined in (13).

We easily get that $\varphi^{(n,k)}u^{(n)} \to \varphi u$ in $L^2(\Omega)$. Therefore, we can apply Proposition 2.1 to prove that the hypotheses of Proposition 7.3 of [4] are satisfied. We thus obtain (26).

This conclude the proof of Proposition 2.4. □

2.2 The upstream weighting scheme

We now define an upstream weighting scheme for the nonlinear problem. Under Hypotheses (2)-(4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 of [4]. We first define the method of upstream weighting of the velocity. Let $\mu > 0$ be given. Let us define

$$\left.\begin{aligned} \gamma_{K,S}^2 &= \sum_{i=1,2} (A_{K,S}^{(i)})^2 (> 0),\\ F_{K,S}^{(v)} &= \sum_{i=1,2} A_{K,S}^{(i)} v_K^{(i)},\\ G_{K,S}^{(v,+)} &= (F_{K,S}^{(v)})^+ + \mu\, \gamma_{K,S},\\ G_{K,S}^{(v,-)} &= (F_{K,S}^{(v)})^- + \mu\, \gamma_{K,S}, \end{aligned}\right\} \text{ for } S \in \mathcal{V}, K \in \mathcal{M}_S \text{ and } v \in H_{\mathcal{D}}(\Omega), \tag{28}$$

and the function $\tilde{v} \in (L_{\mathcal{D}}(\Omega))^2$, depending on v and μ, by:

$$\sum_{K\in\mathcal{M}_S} G_{K,S}^{(v,+)} (v_K^{(k)} - \tilde{v}_S^{(k)}) = 0, \text{ for } k = 1, 2, \forall S \in \mathcal{V}. \tag{29}$$

We may then define the upstream finite volume scheme for the approximation of the solution (1) through the following approximation $b_{\mathcal{D}}$ of the trilinear form b.

$$b_{\mathcal{D}}(u,v,w) = \sum_{K\in\mathcal{M}}\sum_{S\in\mathcal{V}_K}\sum_{k=1,2} w_K^{(k)}\left(G_{K,S}^{(u,+)}v_K^{(k)} - G_{K,S}^{(u,-)}\tilde{v}_S^{(k)}\right),\ \forall(u,v,w)\in(H_{\mathcal{D}}(\Omega))^3. \tag{30}$$

Note that $b_{\mathcal{D}}(u,v,w)$ depends on u through $G_{K,S}^{(u,+)}$ and $G_{K,S}^{(u,-)}$.

The upstream weighting scheme is then defined as the set of equations (12), (28), (29), (30); the existence of a solution to the scheme is stated in Proposition 2.8. Again, we obtain existence by studying a penalized version of the scheme, which may be written as, for a given $\lambda\in(0,+\infty)$: find (u,p) such that

$$\begin{aligned}&(u,p)\in(H_{\mathcal{D}}(\Omega))^2\times L_{\mathcal{D}}(\Omega),\\ &\nu[u,v]_{\mathcal{D}} - \int_\Omega p(x)\mathrm{div}_{\mathcal{D}}(v)(x)dx + b_{\mathcal{D}}(u,u,v) = \int_\Omega f(x)\cdot v(x)dx,\ \forall v\in(H_{\mathcal{D}}(\Omega))^2,\\ &\mathrm{div}_{\mathcal{D}}(u) = -\lambda\ h\ \left(p - \frac{1}{2}\sum_{i=1,2}(\tilde{u}^{(i)})^2\right),\end{aligned} \tag{31}$$

We then have the following estimate.

Proposition 2.5 **[Estimate on the solutions to the penalized upstream scheme]** Under Hypotheses (2)-(4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 of [4]. Let λ and $\mu\in(0,+\infty)$ be given. Let $(u,\tilde{u},p)\in(H_{\mathcal{D}}(\Omega))^2\times(L_{\mathcal{D}}(\Omega))^2\times L_{\mathcal{D}}(\Omega)$ be a solution to the upstream weighting scheme ((28), (29), (30), (31)). Then the following inequalities hold:

$$\nu|u|_{\mathcal{D}} \le \mathrm{diam}(\Omega)\|f\|_{(L^2(\Omega))^2} \tag{32}$$

and

$$(\nu\ \lambda\ h)^{1/2}\left\|p - \frac{1}{2}\sum_{i=1,2}(\tilde{u}^{(i)})^2)\right\|_{L^2(\Omega)} \le \mathrm{diam}(\Omega)\|f\|_{(L^2(\Omega))^2}. \tag{33}$$

Proof Taking $v=u$ in (31), we obtain:

$$\nu|u|_{\mathcal{D}}^2 - \int_\Omega p(x)\mathrm{div}_{\mathcal{D}}(u)(x)dx + b_{\mathcal{D}}(u,u,u) = \int_\Omega f(x)\cdot u(x)dx,$$

which may also be written:

$$\nu|u|_{\mathcal{D}}^2 + \lambda h\int_\Omega p(x)(p(x)-\hat{u}(x))dx + b_{\mathcal{D}}(u,u,u) = \int_\Omega f(x)\cdot u(x)dx, \tag{34}$$

where $\hat{u}\in L_{\mathcal{D}}(\Omega)$ is defined by $\hat{u}_S = \dfrac{1}{2}\sum_{i=1,2}(\tilde{u}_S^{(i)})^2$.

We show in Lemma 2.6 below that for any $u \in (H_{\mathcal{D}}(\Omega))^2$,

$$b_{\mathcal{D}}(u,u,u) \geq \sum_{S\in\mathcal{V}} \sum_{K\in\mathcal{M}_S} \sum_{k=1,2} F_{K,S}^{(u)} \hat{u}_S.$$

Replacing in (34) and noting that the second relation of (31) yields

$$\sum_{K\in\mathcal{M}_S} F_{K,S}^{(u)} = -\lambda\ h\ \mathrm{m}(S)\,(p_S - \hat{u}_S)\,, \forall S\in\mathcal{V},$$

we get

$$\nu|u|_{\mathcal{D}}^2 + \lambda\ h\|p-\hat{u}\|_{L^2(\Omega)}^2 \leq \int_\Omega f(x)\cdot u(x)dx.$$

Applying the discrete Poincaré inequality (see [8] or (20) in [4]) concludes the proof of Proposition 2.5. □

Let us now state and prove the Lemma which we just used.

Lemma 2.6 Under Hypotheses (2)-(4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 of [4]. Let $u \in (H_{\mathcal{D}}(\Omega))^2$ be given. Let $(F_{K,S}^{(u)})_{K\in\mathcal{M},S\in\mathcal{V}}$ and $\tilde{u} \in (L_{\mathcal{D}}(\Omega))^2$ be defined by (28) and (29) (with $v=u$), let $b_{\mathcal{D}}$ be defined in (30), and let $\hat{u} \in L_{\mathcal{D}}(\Omega)$ be given by $\hat{u}_S = \dfrac{1}{2}\sum_{i=1,2}(\tilde{u}_S^{(i)})^2$, for all $S\in\mathcal{V}$. Then

$$b_{\mathcal{D}}(u,u,u) \geq \sum_{S\in\mathcal{V}} \sum_{K\in\mathcal{M}_S} F_{K,S}^{(u)} \hat{u}_S.$$

Proof By definition of $b_{\mathcal{D}}$, one has:

$$b_{\mathcal{D}}(u,u,u) = \sum_{k=1,2} \sum_{K\in\mathcal{M}} \sum_{S\in\mathcal{V}_K} u_K^{(k)} \left(G_{K,S}^{(u,+)} u_K^{(k)} - G_{K,S}^{(u,-)} \tilde{u}_S^{(k)} \right). \tag{35}$$

Since $\sum_{K\in\mathcal{M}_S} G_{K,S}^{(u,+)} \tilde{u}_S^{(k)} (u_K^{(k)} - \tilde{u}_S^{(k)}) = 0$ for all $S\in\mathcal{V}$, we get

$$\sum_{k=1,2} \sum_{S\in\mathcal{V}} \sum_{K\in\mathcal{M}_S} \tilde{u}_S^{(k)} (G_{K,S}^{(u,+)} u_K^{(k)} - G_{K,S}^{(u,-)} \tilde{u}_S^{(k)}) - \sum_{k=1,2} \sum_{S\in\mathcal{V}} \sum_{K\in\mathcal{M}_S} (\tilde{u}_S^{(k)})^2 F_{K,S}^{(u)} = 0.$$

Substracting this equality off (35), one obtains:

$$b_{\mathcal{D}}(u,u,u) = \sum_{k=1,2} \sum_{K\in\mathcal{M}} \sum_{S\in\mathcal{V}_K} \left((u_K^{(k)} - \tilde{u}_S^{(k)}) X_{K,S}^{(k)} + (\tilde{u}_S^{(k)})^2 F_{K,S}^{(u)} \right).$$

with

$$X_{K,S}^{(k)} = G_{K,S}^{(u,+)} u_K^{(k)} - G_{K,S}^{(u,-)} \tilde{u}_S^{(k)}.$$

Writing that

$$\begin{aligned} X_{K,S}^{(k)} &= G_{K,S}^{(u,+)} (u_K^{(k)} - \tilde{u}_S^{(k)}) + \tilde{u}_S^{(k)} (G_{K,S}^{(u,+)} - G_{K,S}^{(u,-)}) \\ &= \frac{1}{2} \Big((G_{K,S}^{(u,+)} + G_{K,S}^{(u,-)})(u_K^{(k)} - \tilde{u}_S^{(k)}) + (G_{K,S}^{(u,+)} - G_{K,S}^{(u,-)})(u_K^{(k)} - \tilde{u}_S^{(k)}) \Big) + \\ &\quad \tilde{u}_S^{(k)} (G_{K,S}^{(u,+)} - G_{K,S}^{(u,-)}) \end{aligned}$$

using the fact that $G^{(u,+)}_{K,S} - G^{(u,-)}_{K,S} = F^{(u)}_{K,S}$ and $G^{(u,+)}_{K,S} + G^{(u,-)}_{K,S} = |F^{(u)}_{K,S}| + 2\mu\gamma_{K,S}$, one obtains:

$$b_{\mathcal{D}}(u,u,u) = \frac{1}{2}\sum_{k=1,2}\sum_{S\in\mathcal{V}}\sum_{K\in\mathcal{M}_S}\left[\begin{array}{l}(u^{(k)}_K - \tilde{u}^{(k)}_S)^2\left(|F^{(u)}_{K,S}| + 2\mu\gamma_{K,S} + F^{(u)}_{K,S}\right) + \\ F^{(u)}_{K,S}(\tilde{u}^{(k)}_S(u^{(k)}_K - \tilde{u}^{(k)}_S) + (u^{(k)}_S)^2)\end{array}\right].$$

Regrouping and using the property

$$\sum_{S\in\mathcal{V}_K} F^{(u)}_{K,S} = \sum_{S\in\mathcal{V}_K}\sum_{i=1,2} A^{(i)}_{K,S}u^{(i)}_K = 0,$$

we get

$$b_{\mathcal{D}}(u,u,u) = \frac{1}{2}\sum_{k=1,2}\sum_{S\in\mathcal{V}}\sum_{K\in\mathcal{M}_S}(u^{(k)}_K - \tilde{u}^{(k)}_S)^2\left(|F^{(u)}_{K,S}| + 2\mu\gamma_{K,S}\right) + \frac{1}{2}\sum_{S\in\mathcal{V}}\sum_{k=1,2}(\tilde{u}^{(k)}_S)^2\sum_{K\in\mathcal{M}_S}F^{(u)}_{K,S}.$$

The first sum of the right hand side is nonnegative, and therefore:

$$b_{\mathcal{D}}(u,u,u) \geq \sum_{S\in\mathcal{V}}\hat{u}_S\sum_{K\in\mathcal{M}_S}F^{(u)}_{K,S}.$$

□

We then deduce the following existence property.

Proposition 2.7 **[Existence of a bounded discrete solution to the penalized upstream weighting scheme]** Under Hypotheses (2)-(4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 of [4]. Let λ and $\mu \in (0,+\infty)$ be given. Then there exists at least one pair $(u,p) \in (H_{\mathcal{D}}(\Omega))^2 \times L_{\mathcal{D}}(\Omega)$, solution to the upstream weighting scheme ((28), (29), (30), (31)). This solution then satisfies (32) and (33).

Proof The proof of Proposition 2.7 is easily obtained by Brouwer's topological degree theorem and estimate (32). Note that the continuity of $\tilde{u}$ as function of u is ensured by the choice $\mu > 0$. □

Proposition 2.8 **[Existence of a bounded discrete solution to the upstream weighting scheme]** Under Hypotheses (2)-(4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 of [4]. Let $\mu \in (0,+\infty)$ be given. Then there exists at least one solution to the scheme ((12), (28), (29), (30)), which verifies (32).

Again, this property is obtained by passing to the limit on a sequence of solutions of ((28), (29), (30), (31)), with values λ tending to 0.

We can now state the following convergence property of the scheme ((28), (29), (30), (31)).

Proposition 2.9 **[Convergence of the penalized upstream weighting scheme in the nonlinear case]** Under Hypotheses (2)-(4), let $\alpha > 0$ be given. Let λ and

$\mu \in (0, +\infty)$ be given and let $(\mathcal{D}^{(n)})_{n\in\mathbb{N}}$ be a sequence of admissible discretizations of Ω in the sense of Definition 2.1 of [4], such that $\lim_{n\to\infty} \text{size}(\mathcal{D}^{(n)}) = 0$ and $\text{angle}(\mathcal{D}^{(n)}) \geq \alpha$, for all $n \in \mathbb{N}$. Let $(u^{(n)}, p^{(n)}) \in (H_{\mathcal{D}^{(n)}}(\Omega))^2 \times L_{\mathcal{D}^{(n)}}(\Omega)$ be a solution to ((28), (29), (30), (31)). Then there exists a subsequence of the sequence $(u^{(n)})_{n\in\mathbb{N}}$ which converges in $L^2(\Omega)^2$ to u, weak solution of the Navier-Stokes problem in the sense of (6).

Proof Again, we follow the proof of Proposition 7.1 in [4]. With the same notations, the only additional property which we need to prove is that:

$$\lim_{n\to\infty} b_{\mathcal{D}^{(n)}}(u^{(n)}, u^{(n)}, \varphi^{(n)}) = b(u, u, \varphi).$$

For the sake of simplicity, we omit for a while the superscripts (n). By definition,

$$b_{\mathcal{D}}(u, u, \varphi) = \sum_{K\in\mathcal{M}} \sum_{S\in\mathcal{V}_K} \sum_{k=1,2} \varphi^{(k)}(x_K)(G_{K,S}^{(u,+)} u_K^{(k)} - G_{K,S}^{(u,-)} \tilde{u}_S^{(k)}), \tag{36}$$

Consider now the following term:

$$T_1 = \sum_{k=1,2} \sum_{K\in\mathcal{M}} \varphi^{(k)}(x_K) \sum_{S\in\mathcal{V}_K} F_{K,S}^{(u)} \tilde{u}_S^{(k)}.$$

As in the case of the centered scheme, we may apply Proposition 2.1 to prove that the hypotheses of Proposition 7.3 of [4] are satisfied. We thus get that

$$\lim_{n\to\infty} T_1^{(n)} = -b(u, u, \varphi).$$

We have

$$\begin{aligned} b_{\mathcal{D}}(u, u, \varphi) - T_1 \quad = \quad & \sum_{k=1,2} \sum_{K\in\mathcal{M}} \varphi(x_K) \sum_{S\in\mathcal{V}_K} \left(G_{K,S}^{(u,+)} u_K^{(k)} - G_{K,S}^{(u,-)} \tilde{u}_S^{(k)} \right) - \\ & \sum_{k=1,2} \sum_{K\in\mathcal{M}} \varphi^{(k)}(x_K) \sum_{S\in\mathcal{V}_K} \tilde{u}_S^{(k)} F_{K,S}^{(u)}. \end{aligned}$$

Since $\sum_{K\in\mathcal{M}_S} G_{K,S}^{(u,+)}(u_K^{(k)} - \tilde{u}_S^{(k)}) = 0$, for $k = 1, 2$, we get

$$\sum_{k=1,2} \sum_{S\in\mathcal{V}} \varphi^{(k)}(x_S) \sum_{K\in\mathcal{M}_S} \left(G_{K,S}^{(u,+)} u_K^{(k)} - G_{K,S}^{(u,-)} \tilde{u}_S^{(k)} \right) - \sum_{k=1,2} \sum_{S\in\mathcal{V}} \varphi^{(k)}(x_S)\, \tilde{u}_S^{(k)} \sum_{K\in\mathcal{M}_S} F_{K,S}^{(u)} = 0.$$

Substracting this last equation off the one above, we get

$$\begin{aligned} b_{\mathcal{D}}(u, u, \varphi) - T_1 \quad = \quad & \sum_{k=1,2} \sum_{S\in\mathcal{V}} \sum_{K\in\mathcal{M}_S} (\varphi^{(k)}(x_K) - \varphi^{(k)}(x_S)) \left(G_{K,S}^{(u,+)} u_K^{(k)} - G_{K,S}^{(u,-)} \tilde{u}_S^{(k)} \right) - \\ & \sum_{k=1,2} \sum_{S\in\mathcal{V}} \sum_{K\in\mathcal{M}_S} (\varphi^{(k)}(x_K) - \varphi^{(k)}(x_S))\, \tilde{u}_S^{(k)} F_{K,S}^{(u)} \\ = \quad & \sum_{k=1,2} \sum_{S\in\mathcal{V}} \sum_{K\in\mathcal{M}_S} (\varphi^{(k)}(x_K) - \varphi^{(k)}(x_S)) G_{K,S}^{(u,-)} (u_K^{(k)} - u_S^{(k)}). \end{aligned}$$

Therefore, $b_{\mathcal{D}}(u,u,\varphi) - T_1 = T_2 + T_3$ with

$$T_2 = \sum_{k=1,2} \sum_{S\in\mathcal{V}} \sum_{K\in\mathcal{M}_S} (\varphi^{(k)}(x_K) - \varphi^{(k)}(x_S)) \left(F^{(u)}_{K,S}\right)^- (u^{(k)}_K - u^{(k)}_S)$$

and

$$T_3 = \mu \sum_{k=1,2} \sum_{S\in\mathcal{V}} \sum_{K\in\mathcal{M}_S} (\varphi(x^{(k)}_K) - \varphi(x^{(k)}_S))\gamma_{K,S}\ (u^{(k)}_K - u^{(k)}_S).$$

Now, by definition of $A^{(i)}_{K,S}$, one has: $\left(F^{(u)}_{K,S}\right)^- \leq \mathrm{diam}(K) \sum_{i=1,2} |u^{(i)}_K|$. Furthermore, thanks to inequalities (12) and (15) of Proposition 2.1 of [4], one has: $\sum_{S\in\mathcal{V}_K} \gamma^2_{K,S} \leq C_2\, C_5\, \mathrm{m}(K)$. Therefore, re-introducing the superscripts (n), one obtains:

$$T^{(n)}_2 \leq hC_9 \sum_{k=1,2} \|u^{(n,k)}\|_{(L^2(\Omega))^2} \sum_{S\in\mathcal{V}} \sum_{K\in\mathcal{M}_S} (u^{(n,k)}_K - u^{(n,k)}_S)^2,$$

which proves, applying Proposition 2.1, that

$$\lim_{n\to\infty} T^{(n)}_2 = 0.$$

On the other hand, we have:

$$|T^{(n)}_3| \leq \mu\ hC_{10} \sum_{k=1,2} \sum_{S\in\mathcal{V}} \sum_{K\in\mathcal{M}_S} (u^{(n,k)}_K - u^{(n,k)}_S)^2,$$

which proves that

$$\lim_{n\to\infty} T^{(n)}_3 = 0.$$

This concludes the proof of the convergence result. □

3 Numerical results

3.1 The lid-driven cavity test

Figure 1 presents the stream lines in the case of the lid-driven cavity flow (see [6]) in the case where the Reynolds number is equal to 400.

Classically, we observe again that for both meshes (unstructured triangles and rectangles) the centered scheme is more precise than the upstream weighting scheme, because of the diffusion which is added by the upstream weighting scheme.

3.2 The backward facing step case

We now study the flow in a backward facing step, for a Reynolds number equal to 800. This is a well documented case in the literature ([1],[2]), and allows to test the performance of methods with respect to the precision on the zones of recirculating flow. Note that we preferred to let a small channel introduce the flow before the step, as in [1], which seems more realistic. The geometrical data of the backward step was also taken from [1]. We computed the streamlines using a reconstruction of

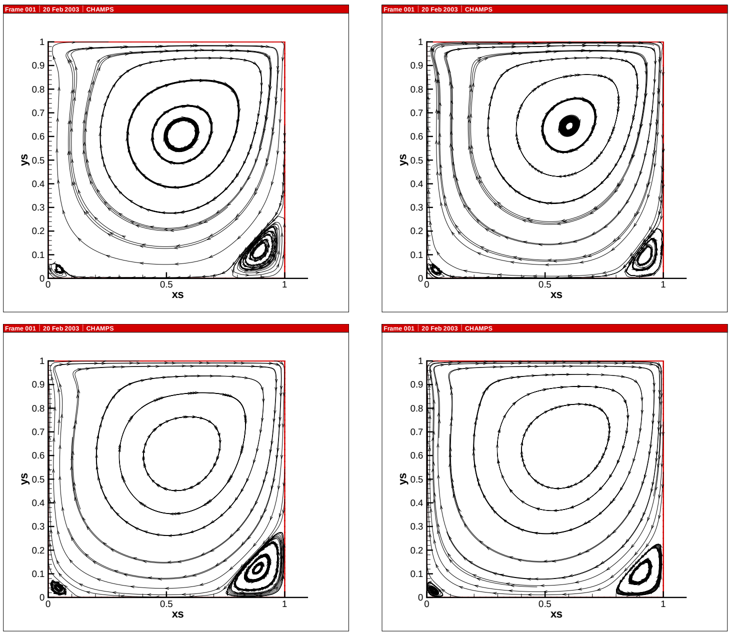

Figure 1: Lid driven cavity, Reynolds = 400, top left: 1400 triangles centered, top right: 1400 triangles upstream weighting, bottom left: 1600 rectangles centered, bottom right: 1600 rectangles upstream weighting.

a discrete potential Φ_σ, located at the edges $\sigma \in \mathcal{E}$ of the mesh. The reconstruction formula is the following:

$$\Phi_{\sigma_1} - \Phi_{\sigma_2} = F^{(u)}_{K,S},$$

where σ_1 and σ_2 are the two edges of $K \in \mathcal{M}$ with the common vertex $S \in \mathcal{V}_K$ ($F^{(u)}_{K,S}$ is defined in (28)).

We present in Figure 2 the streamlines in three different cases: starting form the top, the first figure is obtained with the centered scheme, using a 25200 rectangular grid blocks mesh, the second one with the centered scheme using a 2800 rectangular grid blocks mesh, the third one with the upstream scheme using a 2800 rectangular grid blocks mesh, and the two last ones with respectively the centered and the upstream scheme for 847 cells. It is clear from these figures that the centered scheme is, as one could expect, more precise, but that it becomes unstable for coarser meshes. In fact, for a mesh of 700 cells, the Newton iterations do not converge, even when using an under-relaxation procedure.

In Table 1, we show the separation and reattachement lengths for various schemes referred to in the litterature (see [1, 2]) and for our scheme, in both the centered and upstream cases, and for different grid sizes. The reattachment length of the vortex located on the bottom wall is x_1, the separation and reattachment lengths of the vortex located on the upper wall are denoted x_2 and x_3. In Table 1, we give the results for meshes consisting of 25200, 2800 and 847 cells.

The numerical solution obtained with the centered scheme, using a 25200 rectangular grid blocks mesh seems to be precise enough (comparing the separation and

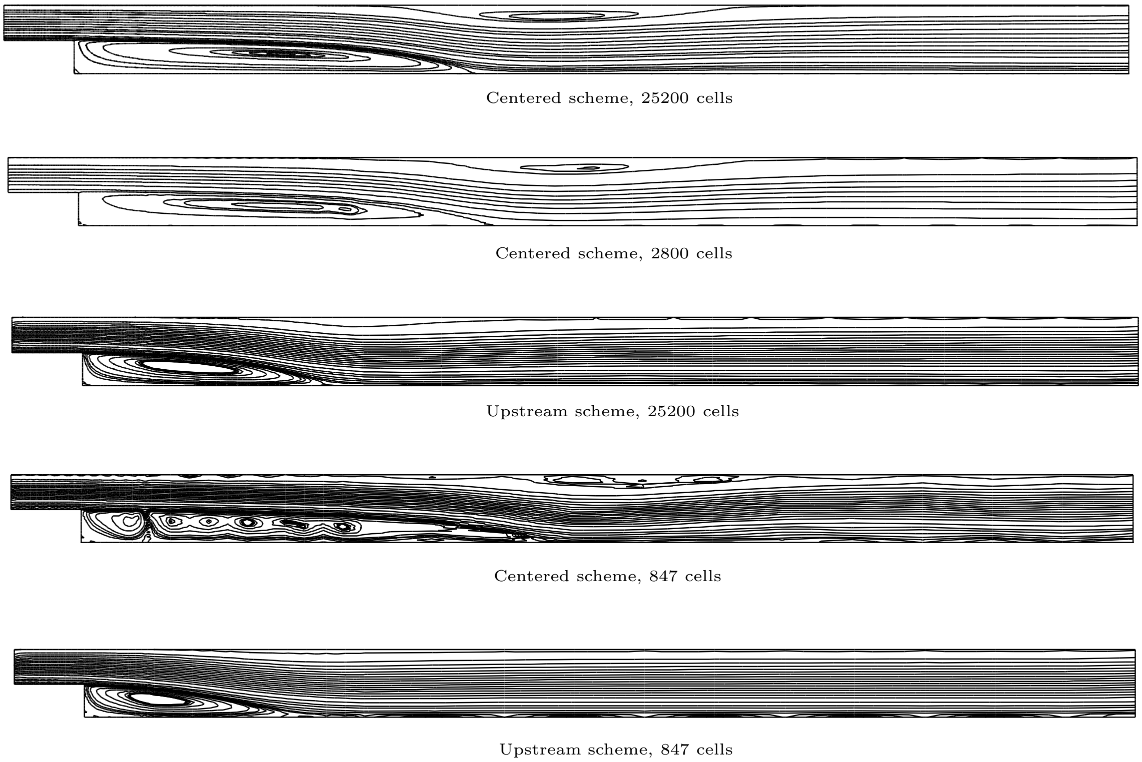

Figure 2: Streamlines for the backward step

reattachment lengths with those of the literature) to be used as a reference solution for experiments carried out on coarser meshes. This allows to compute a rate of convergence of h^2.

We conclude from these numerical tests that the upstream scheme is too diffusive and cannot be used for accurate results, although it has the advantage of remaining stable even on coarse meshes. The centered scheme yields accurate results for a reasonable number of Newton iterations (typically between 5 and 15).

4 Concluding remarks

In this paper we introduced a finite volume scheme for the solution of the stationary Navier-Stokes equations on general meshes in two dimensions. We proved for a penalized version of the scheme.

The observed accuracy of our numerical results indicates that this method is convenient for the calculation of an incompressible flow in two space dimensions. Furthermore, thanks to the location of the discrete unknowns, the finite volume code can be easily coupled with thermal or chemical codes if needed.

Future developments will concentrate on the extension to three-dimensional meshes and to the time-dependent case [5].

	x_1	x_2	x_3
Armaly	14.2	11.2	20.
Sohn	11.5	9.4	18.8
Betts& Sayma	11.21	8.4	20.86
Srinivasan & Rubin	12.44	10.25	20.44
Barton QUICK	12.2	9.64	22.01
Barton SOUBD	12.17	9.61	22.07
Barton SOUD	12.09	9.54	22.21
centered 25200	13.31	11.10	20.64
upstream 25200	10.752	8.70	16.61
centered 2800	13.31	12.22	21.02
upstream 2800	9.12	7.90	12.22
centered 847	16.03	16.03	20.03
upstream 847	7.84	-	-

Table 1: Separation and reattachement lengths, $Re = 800$

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