Convergence analysis of an MPFA method for flow problems in anisotropic heterogeneous porous media

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Abstract

Our purpose in this paper is to present the theoretical analysis of a Multi-Point Flux Approximation method (MPFA method). We start with the derivation of the discrete problem, and then we give a result of existence and uniqueness of a solution for that problem. As in finite element theory, Lagrange interpolation is used to define three classes of continuous and locally polynomial approximate solutions. For analyzing the convergence of these different classes of solutions, the notions of weak and weak-star MPFA approximate solutions are introduced. Their theoretical properties, namely stability and error estimates (in discrete energy norms, $L^2 - norm$ and $L^\infty - norm$), are investigated. These properties play a key role in the analysis (in terms of error estimates for diverse norms) of different classes of continuous and locally polynomial approximate solutions mentioned above.

Key words : diffusion problems, nonhomogeneous anisotropic media, multi-point flux approximation method, weak and weak-star approximate solutions, discrete energy norm, stability, error estimates.

1 Introduction and the model problem

Mixed finite element (MFE) methods have been widely used for modelling flows in porous media as they meet the local and global mass conservation principle and allow an accurate approximation of the velocity on unstructured grids. Note that the local mass conservation and the flux continuity across grid-block interfaces ensure the global mass conservation. The MFE methods also handle well discontinuous coefficients. A computational drawback of these methods is the need to solve an algebraic system of saddle point type. Several methods have been developed in the literature to overcome this issue (see [WY 88] and the references therein). It is well known today that (see for instance [RW 83] and [NMN 06]), in the case of diagonal tensor coefficients and rectangular grids, MFE methods can be reduced to the cell-centered finite volume (CCFV) for the pressure through the use of a suited quadrature rule for the velocity mass matrix. This relationship was explored in [WW 88] to obtain convergence of CCFV on rectangular grids. This result was extended to full tensor coefficients and logically rectangular grids in [AWY 97] and [ADKWY 98], where the so-called expanded mixed finite element method was introduced.

Several other methods have been designed for handling well rough grids and coefficients. In this way, the control volume mixed finite element (CVMFE) method [CJMR 98] is based upon discretizing Darcy's law on specially constructed control volumes. Mimetic finite difference (MFD) methods [HSS 97] are designed to mimic on the discrete level critical properties of the differential operators. The approximating spaces in both methods are closely related to RT_0, the lowest order Raviart-Thomas spaces [RT 77]. These relationships have been explored in [CKK 01] and [BLMS 01][BLSWY] in view to obtain convergence results for the CVMFE methods and the MFD methods, respectively. However, as in the case of MFE methods, both methods lead to an algebraic saddle point problem.

From the outset, the classical finite volume methods (see for instance [BO 04] and [EGH 00]) were designed for ensuring the local mass conservation as well as the robustness for complex applications (multi-phase flow in geologically complex reservoirs for instance). But the imposed geometric constraints to mesh elements was an important handicap for those methods. Furthermore, the finite volume computation of anisotropic flows was a real challenge. To overcome these difficulties, several investigators have proposed challenging finite volume methods. In these methods the key idea consists in approximating the fluxes using multi-point schemes known in the literature as Multi-Point Flux Approximation methods (see for instance [ABBM 94], [ER 94], [ABBM 98], [H 00], [NM 01], [A 02], [E 02], [NN 05], [NM 06] and [CWYM 07]). The Multi-Point Flux Approximation (MPFA) methods combine the advantages of the above mentioned methods, i.e. local and global mass conservation principle, accurate for rough grids and coefficients. The MPFA methods can be considered as finite volume methods of new generation.

According to the literature, one can classify the MPFA methods in three groups: (i) The group made up of flux approximation schemes based upon the pressure values at cell centers and edge mid-points (see [ABBM 94] and [ER 94] who were the pioneers), (ii) The group made up of flux approximation schemes involving the pressure values at cell centers and cell corners (see [H 00] generalizing the ideas developed in [H 98]), (iii) The group made up of flux approximation schemes combining the pressure values at cell centers, cell corners and edge mid-points (see [NM 01][NN 05][NN 06] for quadrilateral grids and [H 03] [NM 06] for unstructured grids). To our knowledge, the MPFA methods from the third group were first developed in [H 03]. Note that some methods very similar to the one proposed in [H 03] and [NN 06] can be found in [CWYM 07] and [BH 07] (see also the references therein).

The convergence analysis of the MPFA methods developed in [ABBM 94] has been carried out in [KW 06a][KW 06b]. The work done in [H 00][H 03] was then reinterpreted in terms of discrete differential operators for isotropic homogeneous diffusion and analyzed in [DO 05] where error estimates can be found. An extension of this idea for general linear and nonlinear diffusion problems has been proposed in [ABH 07] and error analysis has been given therein. A convergence analysis of the MPFA methods proposed in [NN 05] and [NM 06] is developed in [NN 06] for anisotropic diffusion in homogeneous media covered with square grids.

This work is a contribution to the theoretical analysis of the MPFA formulation presented in [NM 01][H 03][NM 06][NN 06]. Our analysis is focused on the case of anisotropic flow in heterogeneous media covered with a square grid. Taking advantage of the computation of the pressures at cell centers, cell corners and edge mid-points allowed by this class of MPFA methods, we introduce and analyze (in terms of error estimates) the concept of locally linear, bilinear and biquadratic approximate pressures following the spirit of the finite element theory [C 78][RT 83].

For presenting our MPFA finite volume formulation, let us consider the 2D diffusion problem consisting in finding a function φ in Ω that satisfies the following partial differential equation associated with homogeneous Dirichlet boundary conditions:

$$-div(D\ grad\ \varphi) = f \qquad \text{in} \quad \Omega \tag{1.1}$$

$$\varphi = 0 \quad \text{on} \qquad \Gamma \tag{1.2}$$

where f is a given function (commonly called source/sink term), Ω is a given open square domain and Γ denotes its boundary. $D = D(x)$, with $x = (x_1, x_2)^t \in \Omega$, is a full symmetric matrix describing the spatial variation of the diffusion coefficient which satisfies the uniform ellipticity i.e.

$$\begin{aligned} &\exists \delta \in \mathbb{R}_+^* \quad \text{such that} \quad \forall \xi \in \mathbb{R}^2,\ \xi \neq 0 \\ &\delta\ |\xi|^2 \leq (\xi)^t\, D(x)\, \xi \qquad a.e. \quad in\ \ \Omega \end{aligned} \tag{1.3}$$

where $()^t$ denotes the transposition operator, $|\ .\ |$ is the euclidian norm in $\mathbb{R}^2$, $D_{ij}(.)$ denotes the components of D which are $L^\infty(\Omega)$-functions.

This paper is organized as follows. The second section deals with a finite volume formulation of the model problem. Within this section we bring an affirmative answer to the well posedness issue concerning the discrete problem. In the third section we introduce various classes of approximate solutions in terms of continuous locally polynomial functions. In the fourth section, we investigate the theoretical properties (stability and error estimates in convenient discrete norms) for the solution of the discrete problem. Based upon the discrete solution properties, convergence results are given via error estimates for different classes of approximate solutions in the fifth section. The sixth section is devoted to conclusions and perspectives of the work.

2 An MPFA formulation of the model problem

Recall that we are interested here in the MPFA formulation proposed in [NM 01][H 03][NN 05][NN 06][NM 06]. Although this formulation applies for general spatially varying full tensor coefficients, we are going to deal with diffusion problems governed by piecewise constant full tensor coefficients. This framework is not restrictive. Indeed, when one deals with the more general case involving tensor coefficients in $L^{\infty}(\Omega)$, one may calculate the mean value of those coefficients in appropriate grid-blocks (i.e. grid-blocks of the primary grid introduced later). From the practical point of view, the assumption of piecewise constant tensor coefficients is very realistic and of common use in practise. Indeed, a subsurface area is made up of a collection of various geologic formations that may be characterized at intermediate scales by averaged full permeability tensors over grid-blocks of the primary grid: for more details on this topic, see [R 05] and [D 05].

One should note that it is naive to think that in real-life problems, all the grid-block interface diffusion coefficients can be described by a regular function (for instance $C^1(\Omega)$-functions). In practice one computes an inter-element coefficient (if necessary) via the homogenization process which involves the coefficients of two adjacent grid-blocks. The computation of inter-element coefficients is a necessary procedure for the MPFA method of Hermeline [H 00] applied to heterogeneous flow problems. But it is not the case for the MPFA methods developed in [NM 01][H 03][NN 05][NM 06][NN 06] where one should introduce (as recommended in [EGH 00]) edge mid-point pressures in the flux computation in view to insure the flux continuity as the diffusion coefficient could be discontinuous over grid-block interfaces. However we should mention that for the MPFA method developed in [NM 01][NM 06][NN 06], the flux continuity over grid-block interfaces is expressed in two ways: (i) the flux continuity is imposed per entire edge for grid-blocks from the primary mesh and per half-edge for grid-blocks from the secondary mesh, (ii) the flux continuity is imposed per entire edge for both primary and secondary grid-blocks. The second way leads to the same discrete problem as the one in [H 03]. To our knowledge, the mathematical analysis (in terms of stability and error estimates in adequate norms, mainly for discontinuous coefficients) of the discrete solutions from MPFA methods in [NM 01][H 03][NN 05][NM 06][NN 06] has not been published.

We intend in this work to focus on the analysis of the discrete solutions from MPFA methods in [NM 01][H 03][NM 06][NN 06], in the case of discontinuous coefficients. We analyze also a concept of continuous and locally polynomial solutions derived from these discrete solutions by Lagrange interpolation theory.

2.1 Formulation of the discrete problem

We present in this subsection the matrix form of our MPFA finite volume formulation for (1.1)-(1.2). Note that this method applies for any convex polygonal domain covered with an unstructured primary grid (see [NM 06]). But we develop here the convergence analysis of that method for a particular domain Ω which is $]0,1[\times]0,1[$. We assume that Ω is covered with a square primary grid denoted $\mathcal{P}$ whose size is $h=\frac{1}{N}$, where N is a given strictly positive integer. On the other hand, we denote K_{ij} the primary grid-block defined by : $K_{ij}=\left[x_1^{i-\frac{1}{2}},x_1^{i+\frac{1}{2}}\right]\times\left[x_2^{j-\frac{1}{2}},x_2^{j+\frac{1}{2}}\right]$ where $x_1^{i+\frac{1}{2}}=x_1^{i-\frac{1}{2}}+h,\ \ x_2^{j+\frac{1}{2}}=x_2^{j-\frac{1}{2}}+h,\ \ for\ i,j=1,\ldots,N$ with $x_1^{\frac{1}{2}}=x_2^{\frac{1}{2}}=0$.

Recall that $L^2(\Omega)$ is the space (of classes) of functions v such that $\int_\Omega v^2dx$ is a finite quantity, and for any positive integer m the so-called Sobolev space $H^m(\Omega)$ is defined by:

$$H^m(\Omega)=\left\{v\in L^2(\Omega)\,;\ \frac{\partial^{|\alpha|}v}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\in L^2(\Omega)\,,\ \text{with}\ \forall 0\leq|\alpha|=\alpha_1+\alpha_2\leq m\right\}$$

where the partial derivatives are taken in the distributional sense. We denote $\|\cdot\|_{m,\Omega}$ the standard norm of $H^m(\Omega)$ and we adopt the convention that $H^0(\Omega)=L^2(\Omega)$, which implies that $\|\cdot\|_{0,\Omega}=\|\cdot\|_{L^2(\Omega)}$.

From the boundary-value problem theory (see for instance [B 83]), the system (1.1)-(1.2) possesses a unique solution in $H^1(\Omega)$ under the assumption (1.3) and the condition $f\in L^2(\Omega)$.

In what follows we assume that the solution φ of (1.1)-(1.2) is sufficiently regular for our purpose (more precisions will be given later about the solution regularity). We should look for a finite volume formulation of the problem (1.1)-(1.2) in terms of a linear system which is derived from the elimination of auxiliary unknowns, namely interface pressures, in flux balance equations over grid-blocks. This linear system involves $\{u_{i,j}\}_{1\leq i,j\leq N}$ and $\left\{u_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\leq i,j\leq N-1}$ as discrete unknowns expected to be reasonable approximations of $\{\varphi_{i,j}\}_{1\leq i,j\leq N}$ (cell center pressures) and $\left\{\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\leq i,j\leq N-1}$ (cell corner pressures) respectively, where $\varphi_{i,j}=\varphi\left(x_1^i,x_2^j\right)$ and $\varphi_{i+\frac{1}{2},j+\frac{1}{2}}=\varphi\left(x_1^{i+\frac{1}{2}},x_2^{j+\frac{1}{2}}\right)$, with:

$$x_1^i=\frac{x_1^{i-\frac{1}{2}}+x_1^{i+\frac{1}{2}}}{2},\qquad x_2^j=\frac{x_2^{j-\frac{1}{2}}+x_2^{j+\frac{1}{2}}}{2}\qquad 1\leq i\,,j\,\leq N \tag{2.1}$$

We also adopt the following conventions:

$$x_1^0=x_1^{\frac{1}{2}},\quad x_1^{N+1}=x_1^{N+\frac{1}{2}},\quad x_2^0=x_2^{\frac{1}{2}},\quad x_2^{N+1}=x_2^{N+\frac{1}{2}} \tag{2.2}$$

We give now a description of the procedure leading to the linear discrete system. Integrating the balance equation (1.1) in the grid-block K_{ij}, centered at the point $\left(x_1^i,x_2^j\right)$ and applying Ostrogradski's theorem leads to integrate the flux on the boundary of K_{ij}. Performing this integration with an adequate quadrature formula over each half-edge of K_{ij} leads to an expression which involves the pressure value at edge mid-points. This pressure is dropped away thanks to the flux continuity which is imposed over the grid-block interfaces.

Let us illustrate now our procedure for computing the fluxes across the grid-block boundaries. For this purpose, we consider the internal edge $[Now, Noe]$ associated with the grid-blocks K_{ij} and K_{ij+1} centered respectively at C and C' (see Figure 1 below).

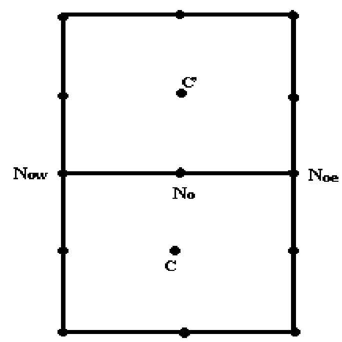

Figure 1: $[Now, Noe]$ is the edge associated with the grid-blocks K_{ij} and K_{ij+1}.

In what follows, the restriction of φ over the closure of each grid-block, denoted again φ, is supposed to be C^2. From the definition of the grid-blocks K_{ij} (see the beginning of this section), it is clear that $(x_1^i,x_2^j)^t$, $(x_1^{i+\frac{1}{2}},x_2^{j+\frac{1}{2}})^t$, $(x_1^{i-\frac{1}{2}},x_2^{j+\frac{1}{2}})^t$ and $(x_1^i,x_2^{j+\frac{1}{2}})^t$ are respectively the coordinates of the points C, Noe, Now and No (see Figure 1 above). On the other hand, we denote D^{ij} and D^{ij+1} respectively the diffusion tensors of the grid-blocks K_{ij} and K_{ij+1}.

The flux expression over the inter-element $[Noe, No]$ viewed as a part of the boundary of the grid-block K_{ij} is given by:

$$\begin{aligned}
\int_{[Noe,No]} -D^{ij} grad\varphi \cdot n \; ds &= -\int_{[Noe,No]} \left[D_{21}^{ij}\frac{\partial \varphi}{\partial x_1} + D_{22}^{ij}\frac{\partial \varphi}{\partial x_2}\right] ds \\
&= -\int_{[Noe,No]} D_{21}^{ij}\frac{\partial \varphi}{\partial x_1} ds - \int_{[Noe,No]} D_{22}^{ij}\frac{\partial \varphi}{\partial x_2} ds \\
&= -D_{21}^{ij}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}} - \varphi_{i,j+\frac{1}{2}}\right] - D_{22}^{ij}\frac{h}{2}\frac{\partial \varphi}{\partial x_2}(No) + R_{ij}^{NoeNo,1}
\end{aligned}$$

where n is the unit outward normal vector and $R_{ij}^{Noe\;No,1} = -h^2 \frac{D_{22}^{ij}}{8} \frac{\partial^2 \varphi}{\partial x_1 \partial x_2}(M)$, with $M \in [No, Noe]$.

Thus we have

$$\int_{[Noe,No]} -D^{ij} grad\varphi \cdot n \; ds = -D_{21}^{ij}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}} - \varphi_{i,j+\frac{1}{2}}\right] + D_{22}^{ij}\left[\varphi_{i,j} - \varphi_{i,j+\frac{1}{2}}\right] + R_{ij}^{NoeNo}$$

where we have set

$$R_{ij}^{NoeNo} = R_{ij}^{NoeNo,1} \quad + \quad h\, R_{ij}^{NoeNo,2}$$

with

$$R_{ij}^{NoeNo,2} = -\,h\frac{D_{22}^{ij}}{8}\frac{\partial^2\varphi}{\partial x_2^2}(Q), \qquad Q \in [C, No]\,.$$

Similarly, we have

$$\int_{[No,Now]} -D^{ij} grad\varphi{\cdot}n\, ds = -D_{21}^{ij}\left[\varphi_{i,j+\frac{1}{2}} - \varphi_{i-\frac{1}{2},j+\frac{1}{2}}\right] + D_{22}^{ij}\left[\varphi_{i,j} - \varphi_{i,j+\frac{1}{2}}\right] + R_{ij}^{NoNow}$$

where

$$R_{ij}^{NoNow} = R_{ij}^{NoNow,1} \quad + \quad h\, R_{ij}^{NoNow,2}$$

with

$$R_{ij}^{NoNow,1} = h^2\frac{D_{22}^{ij}}{8}\frac{\partial^2\varphi}{\partial x_1\partial x_2}(M'), \qquad M' \in [Now, No]$$

and

$$R_{ij}^{NoNow,2} = R_{ij}^{NoeNo,2}$$

It follows from what precedes that the exact flux across the edge $[Noe, Now]$ satisfies to the relation:

$$\begin{aligned}
\int_{[Noe,Now]}\left[-D^{ij} grad\varphi\cdot n\right] ds \;&=\; \int_{[Noe,No]}\left[-D^{ij} grad\varphi\cdot n\right]\, ds \\
&+\; \int_{[No,Now]}\left[-D^{ij} grad\varphi\cdot n\right]\, ds \\
&=\; -D_{21}^{ij}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}} - \varphi_{i,j+\frac{1}{2}}\right] - D_{21}^{ij}\left[\varphi_{i,j+\frac{1}{2}} - \varphi_{i-\frac{1}{2},j+\frac{1}{2}}\right] \\
&+\; 2D_{22}^{ij}\left[\varphi_{i,j} - \varphi_{i,j+\frac{1}{2}}\right] + R_{ij}^{NoeNow}
\end{aligned}$$

where

$$R_{ij}^{NoeNow} = R_{ij}^{NoeNo,1} + R_{ij}^{NoNow,1} + 2\,hR_{ij}^{NoeNo,2}.$$

Since

$$\begin{aligned}
R_{ij}^{NoeNow} \;&=\; h\left[\frac{1}{h}\left(R_{ij}^{NoeNo,1} + R_{ij}^{NoNow,1}\right) + 2R_{ij}^{NoeNo,2}\right] \\
&=\; h\left[h\frac{D_{22}^{ij}}{8}\frac{\partial^2\varphi}{\partial x_1\partial x_2}(M') - h\frac{D_{22}^{ij}}{8}\frac{\partial^2\varphi}{\partial x_1\partial x_2}(M) - h\frac{D_{22}^{ij}}{4}\frac{\partial^2\varphi}{\partial x_2^2}(Q)\right] \\
&=\; h\; R_{i,j}^{e}
\end{aligned}$$

with

$$R_{i,j}^{e} = h\left[\frac{D_{22}^{ij}}{8}\frac{\partial^2\varphi}{\partial x_1\partial x_2}(M') - \frac{D_{22}^{ij}}{8}\frac{\partial^2\varphi}{\partial x_1\partial x_2}(M) - \frac{D_{22}^{ij}}{4}\frac{\partial^2\varphi}{\partial x_2^2}(Q)\right] \tag{2.3}$$

we deduce that

$$\int_{[Noe,Now]}\left[-D^{ij} grad\varphi\cdot n\right] ds = 2D_{22}^{ij}\left[\varphi_{i,j} - \varphi_{i,j+\frac{1}{2}}\right] \;-\; D_{21}^{ij}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}} - \varphi_{i,j+\frac{1}{2}}\right]$$
$$-\,D_{21}^{ij}\left[\varphi_{i,j+\frac{1}{2}} - \varphi_{i-\frac{1}{2},j+\frac{1}{2}}\right] \;+\; h\,R_{i,j}^{e} \tag{2.4}$$

with $\left|R^e_{i,j}\right| \leq C\,h$, where C depends exclusively on Ω, $\frac{\partial^2\varphi}{\partial x_2^2}$ and the lithologic structure of the porous medium.

From (2.4), one naturally approximates the flux across the edge $[Noe, Now]$ as follows:

$$\int_{[Noe,Now]} \left[-D^{ij} grad\varphi \cdot n\right]\, ds \;\approx\; 2\,D^{ij}_{22}\left[\varphi_{i,j} - \varphi_{i,j+\frac{1}{2}}\right] - D^{ij}_{21}\left[\varphi_{i,j+\frac{1}{2}} - \varphi_{i-\frac{1}{2},j+\frac{1}{2}}\right] - D^{ij}_{21}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}} - \varphi_{i,j+\frac{1}{2}}\right] \tag{2.5}$$

REMARK 2.1 When the medium is homogeneous with D as diffusion tensor coefficients, one may suppose (under the boundaries regularity) that the exact pressure φ is in $C^3(\overline{\Omega})$. So the contribution of $\frac{\partial\varphi}{\partial x_2}$ to the flux computation over $[Noe, Now]$ may be performed as follows (via the mid-point rules for integration and derivation):

$$\begin{aligned} -D_{22}\int_{[Noe,Now]} \frac{\partial\varphi}{\partial x_2} ds &= -hD_{22}\frac{\partial\varphi}{\partial x_2}(No) - D_{22}\frac{h^3}{24}\frac{\partial^3\varphi}{\partial x_2^3}(T) \\ &= D_{22}\left[\varphi_{i,j} - \varphi_{i,j+1}\right] + D_{22}\frac{h^3}{24}\left[\frac{\partial^3\varphi}{\partial x_2^3}(S) - \frac{\partial^3\varphi}{\partial x_2^3}(T)\right] \end{aligned}$$

Finally, when the medium is homogeneous (but anisotropic) the flux over $[Noe, Now]$ is given by:

$$\int_{[Noe,Now]} \left[-D\, grad\varphi \cdot n\right] ds = D_{22}\left[\varphi_{i,j} - \varphi_{i,j+1}\right] - D_{21}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}} - \varphi_{i-\frac{1}{2},j+\frac{1}{2}}\right] + E^e_{i,j} \tag{2.6}$$

where

$$E^e_{i,j} = D_{22}\frac{h^3}{24}\left[\frac{\partial^3\varphi}{\partial x_2^3}(S) - \frac{\partial^3\varphi}{\partial x_2^3}(T)\right] \tag{2.7}$$

with $S \in [C, C']$ and $T \in [Noe, Now]$.
This procedure leads to a discrete system involving only approximate pressures at cell centers namely $u_{i,j}$ and cell vertices namely $u_{i+\frac{1}{2},j+\frac{1}{2}}$. In this homogeneous case one can prove that (see [NN 06]): $\varphi_{i,j} - u_{i,j} = O(h^{\frac{3}{2}})$ and $\varphi_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i+\frac{1}{2},j+\frac{1}{2}} = O(h^{\frac{3}{2}})$. However an application of the mid-point rule allows calculating the edge mid-point approximate pressures $u_{i,j+\frac{1}{2}}$ and leads to: $[\varphi_{i,j+\frac{1}{2}} - u_{i,j+\frac{1}{2}}] = O(h^{\frac{3}{2}})$. $\diamondsuit$

For eliminating the edge mid-point pressure $\varphi_{i,j+\frac{1}{2}}$ in the flux approximation formula (2.5), we impose the respect of the flux continuity over the interface between the grid-blocks K_{ij} and K_{ij+1}. Hence we have the following approximation of the flux over the edge $[Noe, Now]$:

$$\int_{[Noe,Now]} \left[-D^{ij}\ grad\,\varphi\ \cdot\ n\right]\ ds\ \approx\ \frac{2D_{22}^{ij}D_{22}^{ij+1}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[\varphi_{i,j}-\varphi_{i,j+1}\right] + \frac{D_{22}^{ij}D_{21}^{ij+1}+D_{22}^{ij+1}D_{21}^{ij}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[\varphi_{i-\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\right]$$

Note that the case of a boundary-edge is performed without any difficulty. Indeed if one deals with a boundary-edge subject to Dirichlet conditions there is no need to impose the flux continuity as the corresponding edge mid-point pressure is given. In the case of a boundary-edge satisfying Neumann conditions, setting that the flux over this edge is equal to the imposed flux leads to an easy elimination of the corresponding edge mid-point pressure.

It is then clear that this procedure applies to the boundary of any grid-block K_{ij}, with $1 \leq\ i,\ j \leq N$, and leads to the following system of relations:

$$\begin{aligned}
&\frac{2D_{22}^{ij}D_{22}^{ij+1}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[\varphi_{i,j}-\varphi_{i,j+1}\right] + \frac{D_{22}^{ij}D_{21}^{ij+1}+D_{22}^{ij+1}D_{21}^{ij}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[\varphi_{i-\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\right]\\
&+\frac{2D_{22}^{ij}D_{22}^{ij-1}}{D_{22}^{ij}+D_{22}^{ij-1}}\left[\varphi_{i,j}-\varphi_{i,j-1}\right] + \frac{D_{22}^{ij}D_{21}^{ij-1}+D_{22}^{ij-1}D_{21}^{ij}}{D_{22}^{ij}+D_{22}^{ij-1}}\left[\varphi_{i+\frac{1}{2},j-\frac{1}{2}}-\varphi_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\
&+\frac{2D_{11}^{ij}D_{11}^{i+1j}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[\varphi_{i,j}-\varphi_{i+1,j}\right] + \frac{D_{11}^{ij}D_{12}^{i+1j}+D_{11}^{i+1j}D_{12}^{ij}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[\varphi_{i+\frac{1}{2},j-\frac{1}{2}}-\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\right]\\
&+\frac{2D_{11}^{ij}D_{11}^{i-1j}}{D_{11}^{ij}+D_{11}^{i-1j}}\left[\varphi_{i,j}-\varphi_{i-1,j}\right] + \frac{D_{11}^{ij}D_{12}^{i-1j}+D_{11}^{i-1j}D_{12}^{ij}}{D_{11}^{ij}+D_{11}^{i-1j}}\left[\varphi_{i-\frac{1}{2},j+\frac{1}{2}}-\varphi_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\
&\approx \int_{K_{ij}} f(x)dx \qquad\qquad \forall\ \ 1 \leq\ i,j \leq N
\end{aligned} \tag{2.8}$$

Note that since the grid-blocks are homogeneous one could integrate straightly over each edge of K_{ij}. Unfortunately this technique does not apply to dual mesh elements introduced later, as they are not homogeneous. From the scientific computing point of view, performing the flux per half-edge for all elements is much more advantageous.

According to the boundary conditions (1.2) and the convention (2.2) we have the following discrete relations:

$$\varphi_{i+\frac{1}{2},\frac{1}{2}} = \varphi_{i+\frac{1}{2},N+\frac{1}{2}} = \varphi_{\frac{1}{2},j+\frac{1}{2}} = \varphi_{N+\frac{1}{2},j+\frac{1}{2}} = 0 \qquad \forall\ \ 0 \leq\ i,j \leq N \tag{2.9}$$

$$\varphi_{i,0} = \varphi_{0,j} = \varphi_{i,N+1} = \varphi_{N+1,j} = 0 \qquad \forall\ \ 1 \leq\ i,j \leq N \tag{2.10}$$

The discrete system (2.8)-(2.10) is not closed since the number of unknowns is greater than the number of equations. Indeed there are $[N^2 + (N-1)^2 + 8N]$ unknowns and only $[N^2+8N]$ equations. Therefore we should add $(N-1)^2$ equations to that system.

For this purpose, it is natural to integrate the balance equation (1.1) in the finite volume $K_{i+\frac{1}{2},j+\frac{1}{2}} = \left[x_1^i, x_1^{i+1}\right] \times \left[x_2^j, x_2^{j+1}\right]$. Applying once more Ostrogradski's theorem and a quadrature formula for approximating the flux on the boundary of $K_{i+\frac{1}{2},j+\frac{1}{2}}$ leads to the following relation:

$$\begin{aligned}
&\tfrac{D_{11}^{ij+1}D_{21}^{i+1j+1}+D_{11}^{i+1j+1}D_{21}^{ij+1}}{D_{11}^{ij+1}+D_{11}^{i+1j+1}}\left[\varphi_{i,j+1}-\varphi_{i+1,j+1}\right]+\\
&\left(\tfrac{\left(D_{12}^{i+1j+1}-D_{12}^{ij+1}\right)\left(D_{21}^{ij+1}-D_{21}^{i+1j+1}\right)}{2\left(D_{11}^{ij+1}+D_{11}^{i+1j+1}\right)}+\tfrac{D_{22}^{ij+1}+D_{22}^{i+1j+1}}{2}\right)\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j+\frac{3}{2}}\right]\\
&+\tfrac{D_{11}^{i+1j}D_{21}^{ij}+D_{11}^{ij}D_{21}^{i+1j}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[\varphi_{i+1,j}-\varphi_{i,j}\right]+\\
&\left(\tfrac{\left(D_{12}^{i+1j}-D_{12}^{ij}\right)\left(D_{21}^{ij}-D_{21}^{i+1j}\right)}{2\left(D_{11}^{ij}+D_{11}^{i+1j}\right)}+\tfrac{D_{22}^{ij}+D_{22}^{i+1j}}{2}\right)\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j-\frac{1}{2}}\right]\\
&+\tfrac{D_{22}^{i+1j}D_{12}^{i+1j+1}+D_{22}^{i+1j+1}D_{12}^{i+1j}}{D_{22}^{i+1j}+D_{22}^{i+1j+1}}\left[\varphi_{i+1,j}-\varphi_{i+1,j+1}\right]+\\
&\left(\tfrac{\left(D_{21}^{i+1j+1}-D_{21}^{i+1j}\right)\left(D_{12}^{i+1j}-D_{12}^{i+1j+1}\right)}{2\left(D_{22}^{i+1j}+D_{22}^{i+1j+1}\right)}+\tfrac{D_{11}^{i+1j}+D_{11}^{i+1j+1}}{2}\right)\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{3}{2},j+\frac{1}{2}}\right]\\
&+\tfrac{D_{22}^{ij+1}D_{12}^{ij}+D_{22}^{ij}D_{12}^{ij+1}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[\varphi_{i,j+1}-\varphi_{i,j}\right]+\\
&\left(\tfrac{\left(D_{21}^{ij+1}-D_{21}^{ij}\right)\left(D_{12}^{ij}-D_{12}^{ij+1}\right)}{2\left(D_{22}^{ij}+D_{22}^{ij+1}\right)}+\tfrac{D_{11}^{ij}+D_{11}^{ij+1}}{2}\right)\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i-\frac{1}{2},j+\frac{1}{2}}\right]\\
&\approx\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}f(x)dx\qquad\forall 1\le i,j\le N-1
\end{aligned}\tag{2.11}$$

Note that the exact solution φ does not satisfy (2.8)-(2.11) with equalities everywhere (see (2.8) and (2.11)). We derive the discrete system from (2.8)-(2.11) replacing φ and " $\approx$ " by u and " $=$ " respectively. Therefore the discrete problem consists in finding $\{u_{i,j}\}_{1\le i,j\le N}$ and $\left\{u_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\le i,j\le N-1}$ real unknowns such that:

$$\begin{aligned}
&\tfrac{2D_{22}^{ij}D_{22}^{ij+1}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[u_{i,j}-u_{i,j+1}\right]+\tfrac{D_{22}^{ij}D_{21}^{ij+1}+D_{22}^{ij+1}D_{21}^{ij}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[u_{i-\frac{1}{2},j+\frac{1}{2}}-u_{i+\frac{1}{2},j+\frac{1}{2}}\right]\\
&+\tfrac{2D_{22}^{ij}D_{22}^{ij-1}}{D_{22}^{ij}+D_{22}^{ij-1}}\left[u_{i,j}-u_{i,j-1}\right]+\tfrac{D_{22}^{ij}D_{21}^{ij-1}+D_{22}^{ij-1}D_{21}^{ij}}{D_{22}^{ij}+D_{22}^{ij-1}}\left[u_{i+\frac{1}{2},j-\frac{1}{2}}-u_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\
&+\tfrac{2D_{11}^{ij}D_{11}^{i+1j}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[u_{i,j}-u_{i+1,j}\right]+\tfrac{D_{11}^{ij}D_{12}^{i+1j}+D_{11}^{i+1j}D_{12}^{ij}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[u_{i+\frac{1}{2},j-\frac{1}{2}}-u_{i+\frac{1}{2},j+\frac{1}{2}}\right]\\
&+\tfrac{2D_{11}^{ij}D_{11}^{i-1j}}{D_{11}^{ij}+D_{11}^{i-1j}}\left[u_{i,j}-u_{i-1,j}\right]+\tfrac{D_{11}^{ij}D_{12}^{i-1j}+D_{11}^{i-1j}D_{12}^{ij}}{D_{11}^{ij}+D_{11}^{i-1j}}\left[u_{i-\frac{1}{2},j+\frac{1}{2}}-u_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\
&=\int_{K_{ij}}f(x)dx\qquad\forall 1\le i,j\le N
\end{aligned}\tag{2.12}$$

$$
\begin{aligned}
&\tfrac{D_{11}^{ij+1}D_{21}^{i+1j+1}+D_{11}^{i+1j+1}D_{21}^{ij+1}}{D_{11}^{ij+1}+D_{11}^{i+1j+1}}\left[u_{i,j+1}-u_{i+1,j+1}\right]+\\
&\left(\tfrac{\left(D_{12}^{i+1j+1}-D_{12}^{ij+1}\right)\left(D_{21}^{ij+1}-D_{21}^{i+1j+1}\right)}{2\left(D_{11}^{ij+1}+D_{11}^{i+1j+1}\right)}+\tfrac{D_{22}^{ij+1}+D_{22}^{i+1j+1}}{2}\right)\left[u_{i+\frac{1}{2},j+\frac{1}{2}}-u_{i+\frac{1}{2},j+\frac{3}{2}}\right]\\
&+\tfrac{D_{11}^{i+1j}D_{21}^{ij}+D_{11}^{ij}D_{21}^{i+1j}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[u_{i+1,j}-u_{i,j}\right]+\\
&\left(\tfrac{\left(D_{12}^{i+1j}-D_{12}^{ij}\right)\left(D_{21}^{ij}-D_{21}^{i+1j}\right)}{2\left(D_{11}^{ij}+D_{11}^{i+1j}\right)}+\tfrac{D_{22}^{ij}+D_{22}^{i+1j}}{2}\right)\left[u_{i+\frac{1}{2},j+\frac{1}{2}}-u_{i+\frac{1}{2},j-\frac{1}{2}}\right]\\
&+\tfrac{D_{22}^{i+1j}D_{12}^{i+1j+1}+D_{22}^{i+1j+1}D_{12}^{i+1j}}{D_{22}^{i+1j}+D_{22}^{i+1j+1}}\left[u_{i+1,j}-u_{i+1,j+1}\right]+ \qquad (2.13)\\
&\left(\tfrac{\left(D_{21}^{i+1j+1}-D_{21}^{i+1j}\right)\left(D_{12}^{i+1j}-D_{12}^{i+1j+1}\right)}{2\left(D_{22}^{i+1j}+D_{22}^{i+1j+1}\right)}+\tfrac{D_{11}^{i+1j}+D_{11}^{i+1j+1}}{2}\right)\left[u_{i+\frac{1}{2},j+\frac{1}{2}}-u_{i+\frac{3}{2},j+\frac{1}{2}}\right]\\
&+\tfrac{D_{22}^{ij+1}D_{12}^{ij}+D_{22}^{ij}D_{12}^{ij+1}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[u_{i,j+1}-u_{i,j}\right]+\\
&\left(\tfrac{\left(D_{21}^{ij+1}-D_{21}^{ij}\right)\left(D_{12}^{ij}-D_{12}^{ij+1}\right)}{2\left(D_{22}^{ij}+D_{22}^{ij+1}\right)}+\tfrac{D_{11}^{ij}+D_{11}^{ij+1}}{2}\right)\left[u_{i+\frac{1}{2},j+\frac{1}{2}}-u_{i-\frac{1}{2},j+\frac{1}{2}}\right]\\
&=\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}f(x)dx \qquad \forall 1\le i,j\le N-1
\end{aligned}
$$

where we have set

$$u_{i+\frac{1}{2},\frac{1}{2}}=u_{i+\frac{1}{2},N+\frac{1}{2}}=u_{\frac{1}{2},j+\frac{1}{2}}=u_{N+\frac{1}{2},j+\frac{1}{2}}=0 \qquad \forall 0\le i,j\le N \qquad (2.14)$$

and

$$u_{i,0}=u_{0,j}=u_{i,N+1}=u_{N+1,j}=0 \qquad \forall 1\le i,j\le N \qquad (2.15)$$

Note that this MPFA method applies to unstructured meshes covering any polygonal domain as shown in [NM 06]. The theoretical analysis of this finite volume method remains an open problem. However one can find in [NN 06] a convergence analysis of this MPFA method in the case of diffusion in homogeneous media covered with a square grid and subject to Dirichlet conditions.

If the preceding discrete problem gets a unique solution, one can deduce the approximate values of the pressure at the edge mid-points using the following relations which expresses the flux continuity at grid-block interfaces.
For $\quad 0\le i\le N \quad$ and $\quad 1\le j\le N$:

$$
\begin{aligned}
u_{i+\frac{1}{2},j}=\tfrac{1}{2\left(D_{11}^{i,j}+D_{11}^{i+1,j}\right)}&\Big\{2D_{11}^{i,j}u_{ij}+2D_{11}^{i+1,j}u_{i+1,j}\\
&+\left[D_{12}^{i+1,j}-D_{12}^{i,j}\right]\left[u_{i+\frac{1}{2},j+\frac{1}{2}}-u_{i+\frac{1}{2},j-\frac{1}{2}}\right]\Big\}
\end{aligned}
\qquad (2.16)
$$

For $\quad 1 \leq i \leq N \quad$ and $\quad 0 \leq j \leq N$:

$$u_{i,j+\frac{1}{2}} = \frac{1}{2\left(D_{22}^{ij}+D_{22}^{ij+1}\right)} \Big\{ 2D_{22}^{ij} u_{i,j} + 2D_{22}^{ij+1} u_{i,j+1} + \left[D_{21}^{ij+1} - D_{21}^{ij}\right] \left[u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i-\frac{1}{2},j+\frac{1}{2}}\right] \Big\} \tag{2.17}$$

Therefore one can deduce the fluxes over the grid-block interfaces from the following relations:

$$q_{i+\frac{1}{2},j} = D_{12}^{ij} \left[u_{i+\frac{1}{2},j-\frac{1}{2}} - u_{i+\frac{1}{2},j}\right] + 2D_{11}^{ij} \left[u_{i,j} - u_{i+\frac{1}{2},j}\right] + D_{12}^{ij} \left[u_{i+\frac{1}{2},j} - u_{i+\frac{1}{2},j+\frac{1}{2}}\right] \tag{2.18}$$

$$q_{i,j+\frac{1}{2}} = -D_{21}^{ij} \left[u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i,j+\frac{1}{2}}\right] + 2D_{22}^{ij} \left[u_{i,j} - u_{i,j+\frac{1}{2}}\right] - D_{21}^{ij} \left[u_{i,j+\frac{1}{2}} - u_{i-\frac{1}{2},j+\frac{1}{2}}\right] \tag{2.19}$$

2.2 Existence and uniqueness for a solution of the discrete problem

We are going to deal now with the existence and uniqueness of a solution for the discrete problem (2.12)-(2.13). Before giving the two main results of this subsection, let us shortly comment about this discrete problem. Its matrix form may be expressed as follows:

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} U_{cc} \\ U_{vc} \end{pmatrix} = \begin{pmatrix} F_{cc} \\ F_{vc} \end{pmatrix} \tag{2.20}$$

where we have set :

$$U_{cc} = \{u_{i,j}\}_{1\leq i,j \leq N} \qquad \text{and} \qquad U_{vc} = \left\{u_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\leq i,j\leq N-1} \tag{2.21}$$

and where:

F_{cc} is a sub-vector with N^2 components defined by the right hand side of (2.12) only as we account with (2.14) and (2.15).

F_{vc} is a sub-vector with $(N-1)^2$ components defined by the right hand side of (2.13) only as we account with (2.14) and (2.15).

A is a $N \times N$ symmetric positive definite matrix, associated to the classical grid-centered finite volume when D is diagonal i.e.$D_{12} = D_{21} = 0$.

C is a $(N-1) \times (N-1)$ symmetric positive definite matrix, associated to the classical vertex-centered finite volume when D is diagonal.

B is a $N \times (N-1)$ matrix and B^T is its transpose.

When the diffusion coefficient D is reduced to a diagonal matrix, the discrete problem (2.12)-(2.13) admits the following matrix form

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} U_{cc} \\ U_{vc} \end{pmatrix} = \begin{pmatrix} F_{cc} \\ F_{vc} \end{pmatrix} \tag{2.22}$$

When solving this system on a parallel computer using two processors, the CPU time is almost the same as when solving only one of the two following subsystems :

$$A\,U_{cc} = F_{cc} \quad \text{and} \quad C\,U_{vc} = F_{vc}$$

Since A and C are both positive definite, the existence and the uniqueness for a solution of (2.22) are ensured. Therefore when D is diagonal, our formulation provides more information about the solution of the diffusion problem than the classical finite volume formulations, for almost the same CPU time on a parallel computer equipped with two processors.

Let us give now the two main results of this subsection.

Proposition 2.2 The discrete problem consisting to find $\{u_{i,j}\}_{1\le i,j\le N}$ and $\left\{u_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\le i,j\le N-1}$ such that the equations (2.12)-(2.13) are satisfied under the conditions (2.14) and (2.15), possesses a unique solution. $\diamond$

Proposition 2.3 The matrix $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ associated to the discrete problem (2.12)-(2.13) is symmetric and positive definite. $\diamond$

Since Proposition 2.3 implies Proposition 2.2, let us focus on the proof of the last proposition.

Proof. The symmetric structure of the matrix $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ is obvious from the comments located between (2.21) and (2.22). Multiplying (2.12) by $u_{i,j}$ and (2.13) by $u_{i+\frac{1}{2},j+\frac{1}{2}}$ and summing leads to (here notations (2.21) are utilized) :

$$[U_{cc}\ U_{vc}] \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} U_{cc} \\ U_{vc} \end{bmatrix} = RHS1 + RHS2$$

where

$$\begin{aligned} RHS1 = \sum_{\substack{1\le i\le N \\ 0\le j\le N}} \Big\{ & D_{11}^{ij,ij+1} \left(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i-\frac{1}{2},j+\frac{1}{2}}\right)^2 + D_{22}^{ij,ij+1} \left(u_{i,j+1} - u_{i,j}\right)^2 \\ + \quad & 2D_{12}^{ij,ij+1} \left(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i-\frac{1}{2},j+\frac{1}{2}}\right) \left(u_{i,j+1} - u_{i,j}\right) \Big\} \end{aligned}$$

$$\begin{aligned} RHS2 = \sum_{\substack{0\le i\le N \\ 1\le j\le N}} \Big\{ & D_{11}^{ij,i+1j} \left(u_{i+1,j} - u_{i,j}\right)^2 + D_{22}^{ij,i+1j} \left(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i+\frac{1}{2},j-\frac{1}{2}}\right)^2 \\ + \quad & 2\,D_{21}^{ij,i+1j} \left(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i+\frac{1}{2},j-\frac{1}{2}}\right) \left(u_{i+1,j} - u_{i,j}\right) \Big\} \end{aligned}$$

where we have set

$$
\begin{aligned}
D_{11}^{ij,ij+1} &= -\frac{\left(D_{12}^{ij+1}-D_{12}^{ij}\right)^2}{2\left(D_{22}^{ij}+D_{22}^{ij+1}\right)} + \frac{D_{11}^{ij}+D_{11}^{ij+1}}{2} \\
D_{22}^{ij,ij+1} &= \frac{2D_{22}^{ij}D_{22}^{ij+1}}{D_{22}^{ij}+D_{22}^{ij+1}}, \qquad D_{21}^{ij,ij+1} = \frac{D_{22}^{ij}D_{21}^{ij+1}+D_{22}^{ij+1}D_{21}^{ij}}{D_{22}^{ij}+D_{22}^{ij+1}}
\end{aligned}
\tag{2.23}
$$

$$
\begin{aligned}
D_{11}^{ij,i+1j} &= \frac{2D_{11}^{ij}D_{11}^{i+1j}}{D_{11}^{ij}+D_{11}^{i+1j}}, \qquad D_{21}^{ij,i+1j} = \frac{D_{11}^{ij}D_{21}^{i+1j}+D_{11}^{i+1j}D_{21}^{ij}}{D_{11}^{ij}+D_{11}^{i+1j}} \\
D_{22}^{ij,i+1j} &= -\frac{\left(D_{21}^{i+1j}-D_{21}^{ij}\right)^2}{2\left(D_{11}^{ij}+D_{11}^{i+1j}\right)} + \frac{D_{22}^{ij}+D_{22}^{i+1j}}{2}
\end{aligned}
\tag{2.24}
$$

It is clear that $D^{ij,ij+1}$ and $D^{ij,i+1j}$ are symmetric since $D(x)$ is supposed to be symmetric. Moreover, one can easily check that $D^{ij,ij+1}$ and $D^{ij,i+1j}$ are positive definite. Therefore these matrices possess strictly positive eigenvalues. Let $\lambda_{\min}^{ij,ij+1}$ and $\lambda_{\min}^{ij,i+1j}$ be respectively their least eigenvalues. So we have

$$
RHS1 \geq \sum_{\substack{1\leq i\leq N\\0\leq j\leq N}} \lambda_{\min}^{ij,ij+1} \left\{\left(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i-\frac{1}{2},j+\frac{1}{2}}\right)^2 + \left(u_{i,j+1}-u_{i,j}\right)^2\right\}
$$

$$
RHS2 \geq \sum_{\substack{0\leq i\leq N\\1\leq j\leq N}} \lambda_{\min}^{ij,i+1j} \left\{\left(u_{i+1,j}-u_{i,j}\right)^2 + \left(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i+\frac{1}{2},j-\frac{1}{2}}\right)^2\right\}
$$

Therefore

$$
\begin{aligned}
RHS1 + RHS2 \geq & \sum_{\substack{1\leq i\leq N\\0\leq j\leq N}} \lambda_{\min}^{ij,ij+1} \left\{\left(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i-\frac{1}{2},j+\frac{1}{2}}\right)^2 + \left(u_{i,j+1}-u_{i,j}\right)^2\right\} \\
& + \sum_{\substack{0\leq i\leq N\\1\leq j\leq N}} \lambda_{\min}^{ij,i+1j} \left\{\left(u_{i+1,j}-u_{i,j}\right)^2 + \left(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i+\frac{1}{2},j-\frac{1}{2}}\right)^2\right\}
\end{aligned}
\tag{2.25}
$$

Thanks to the relations (2.14) and (2.15) the equality holds if and only if $U_{cc} = 0$ and $U_{vc} = 0$. Thus the positive definiteness of the matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ is proven. Therefore, it is clear that the discrete problem which consists in finding $\{u_{i,j}\}_{1\leq i,j\leq N}$ and $\left\{u_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\leq i,j\leq N-1}$ such that the equations (2.12)-(2.13) are satisfied possesses a unique solution ■

REMARK 2.4 It follows from the MPFA formulation (2.12)-(2.15) that the effective permeabilities for the grid-blocks $(i,j)-(i,j+1)$ and $(i,j)-(i+1,j)$ are respectively given by (2.23) and (2.24). These new formulas generalize well known algebraic formulas for equivalent permeabilities in heterogeneous isotropic porous media [R 05]. For more information on this topic one can see [D 05] and [R 05]. ♢

3 Introduction of three classes of approximate solutions in terms of continuous functions

Solving the discrete problem (2.12)-(2.13) leads to determining all the discrete unknowns at grid-block centers and grid-block corners (with respect to the primary grid). Then we deduce the approximate pressures at edge mid-points via (2.16) and (2.17). In what follows a node is a grid-block center or a grid-block corner or an edge mid-point. On the other hand, we make use of the simplified notation u_m representing either $u_{i,j}$, $u_{i+\frac{1}{2},j+\frac{1}{2}}$, $u_{i+\frac{1}{2},j}$ or $u_{i,j+\frac{1}{2}}$ which are nodal values.

3.1 The class of piecewise linear approximate solutions

We start by dividing each grid-block (of the primary grid $\mathcal{P}$) into four triangular elements with generic name T constructed by joining each grid-block center to the four corresponding grid-block corners (see Figure 2 below). By doing so, one generates over Ω a new grid denoted $\mathcal{T}$. Let us denote $U^h_{\mathcal{T}}$ the piecewise linear approximate solution associated with the grid $\mathcal{T}$. The quantities u_m actually correspond here to the values of $U^h_{\mathcal{T}}$ at grid-block centers and grid-block corners. Thus these quantities satisfy the following equality:

$$u_m = U^h_{\mathcal{T}}(x^{(m)}), \text{where } x^{(m)} \text{ is a grid-block center or a grid-block corner.}$$

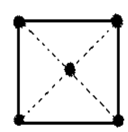

Figure 2: A (primary) grid block divided into four triangular elements T for a piece-wise linear approximation of the solution. The symbol • represents a degree of freedom (which is a nodal value) of the approximate solution over triangular elements.

Definition 3.1 Let $x^{(i)}$, $x^{(j)}$ and $x^{(k)}$ denote the vertices of a triangular element $T \in \mathcal{T}$. The approximate solution $U^h_{\mathcal{T}}$ of the diffusion problem (1.1)-(1.2) is defined in T as follows:

$$U^h_{\mathcal{T}}(x) = \alpha \cdot (x - x^{(i)}) + u_i$$

where $x = (x_1, x_2)^t$, $\alpha = (\alpha_1, \alpha_2)^t$, $x^{(i)} = (x_1^{(i)}, x_2^{(i)})^t$ and $u_i = U^h_{\mathcal{T}}(x^{(i)})$. The components of the vector α are easily calculated due to the fact that $u_j = U^h_{\mathcal{T}}(x^{(j)})$ and $u_k = U^h_{\mathcal{T}}(x^{(k)})$ are data available from the solution of the discrete problem (2.12)-(2.13). ◊

We have the following obvious result.

Proposition 3.2 [illegible] $U^h_{\mathcal{T}}$ [illegible] $\overline{\Omega}$ [illegible] Ω) [illegible] $U^h_{\mathcal{T}}$ [illegible] $H^1_0(\Omega)$. ◊

Recall that $H^1_0(\Omega)$ is defined as follows:

$$H^1_0(\Omega) = \left\{ v \in H^1(\Omega); \quad v = 0 \quad \text{on} \quad \Gamma \ \right\} \tag{3.1}$$

and the mapping

$$v \longmapsto \left[\int_\Omega |grad\ v|^2\, dx \right]^{\frac{1}{2}} \tag{3.2}$$

defines the well-known $H^1_0(\Omega) - norm$.

3.2 The class of piecewise bilinear approximate solutions

We start by dividing each square grid-block (of the primary grid) into four square elements with generic name S (see Figure 3 below). By doing so, one generates over Ω a new grid denoted $\mathcal{S}$. Let us denote $U^h_{\mathcal{S}}$ the piecewise bilinear approximate solution associated with the grid $\mathcal{S}$. The quantities u_m actually correspond here to the values of $U^h_{\mathcal{S}}$ at grid-block centers, grid-block corners and edge mid-points. Thus these quantities satisfy the relation
$u_m = U^h_{\mathcal{S}}(x^{(m)})$, where $x^{(m)}$ is a node.

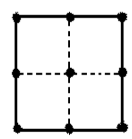

Figure 3: A (primary) grid block divided into four square elements S for a piecewise bilinear approximation of the solution. The symbol $\bullet$ represents a degree of freedom (which is a nodal value) of the approximate solution over square elements.

DEFINITION 3.3 Let $x^{(i)}$, $x^{(j)}$, $x^{(k)}$ and $x^{(l)}$ denote the vertices of a square element $S \in \mathcal{S}$. The approximate solution $U^h_{\mathcal{S}}$ of the diffusion problem (1.1)-(1.2) is defined in S as follows:

$$U^h_{\mathcal{S}}(x) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2$$

where $x = (x_1, x_2)^t$ and where $a_{00}, a_{10}, a_{01}, a_{11}$ are calculated thanks to the quantities u_i, u_j, u_k and u_l given by the solution of the discrete problem (2.12)-(2.13) and the relations (2.16)-(2.17). $\Diamond$

PROPOSITION 3.4 [illegible] $U^h_{\mathcal{S}}$ [illegible] $\overline{\Omega}$. [illegible] $U^h_{\mathcal{S}}$ [illegible] $H^1_0(\Omega)$ $\Diamond$

Proof.

To fix the ideas, we set (see Figure 4):
$U^h_{\mathcal{S}|E_1}(x_1, x_2) = a + bx_1 + cx_2 + dx_1x_2, \quad U^h_{\mathcal{S}|E_2}(x_1, x_2) = e + fx_1 + gx_2 + kx_1x_2$
Over the interface [A,B] of the square elements E_1 and E_2, the function $U^h_{\mathcal{S}|E_1} - U^h_{\mathcal{S}|E_2}$

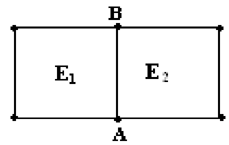

Figure 4: Two adjacent square elements of the grid $\mathcal{S}$ with a vertical interface [A,B].

depends exclusively on x_2, since x_1 is a given constant equal to x_1^A (the coordinate of A in the first direction). So $\left(U^h_{\mathcal{S}|E_1} - U^h_{\mathcal{S}|E_2}\right)$ reduces to a linear function $P(t) = \alpha\, t + \beta$, where we have set (of course) $t = x_2$ and where α and β are given constants. Since $P\left(x_2^A\right) = P\left(x_2^B\right) = 0$, P is identically null on the interface [A,B]. Thus $U^h_{\mathcal{S}}$ is continuous over [A,B].
Nearly one may use the same arguments in the case of an horizontal interface. The proof of Proposition 3.4 is ended ■

3.3 The class of piecewise biquadratic approximate solutions

Let us denote $U^h_{\mathcal{P}}$ the piecewise biquadratic approximate solution associated with the primary grid $\mathcal{P}$. The computed quantities u_m correspond here to the values of $U^h_{\mathcal{P}}$ at grid-block centers, grid-block corners and edge mid-points of the primary grid. Thus these quantities satisfy the following relation:
$u_m = U^h_{\mathcal{P}}(x^{(m)})$, where $x^{(m)}$ is a node (see Figure 5 below).

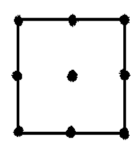

Figure 5: The symbol • represents a degree of freedom (which is a nodal value) of the biquadratic approximate solution over a primary grid-block.

Definition 3.5 Let $x^{(i)}$, $x^{(j)}$, $x^{(k)}$ and $x^{(l)}$ denote the vertices of a square element $E \in \mathcal{P}$. The approximate solution $U^h_{\mathcal{P}}$ of the diffusion problem (1.1)-(1.2) is defined in E as follows:

$$U^h_{\mathcal{P}}(x) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2 +$$

$$a_{20}x_1^2 + a_{02}x_2^2 + a_{21}x_1^2x_2 + a_{12}x_1x_2^2 + a_{22}x_1^2x_2^2$$

where $x = (x_1, x_2)^t$ and where $a_{00}, a_{10}, a_{01}, a_{11}, a_{20}, a_{02}, a_{21}, a_{12}, a_{22}$ are calculated thanks to the quantities u_m corresponding to the values of $U^h_{\mathcal{P}}$ at the center of E, corners of E and mid-points of E-edges given by the solution of the discrete problem (2.12)-(2.13) and the relations (2.16)-(2.17). $\diamondsuit$

PROPOSITION 3.6 [illegible] $U^h_{\mathcal{P}}$ [illegible] $\overline{\Omega}$ [illegible] Ω) [illegible] $U^h_{\mathcal{P}}$ [illegible] $H^1_0(\Omega)$. ◊

Proof. One may use almost the same arguments as in the proof of Proposition 3.4. ■

4 Stability and error estimates for the solution of the discrete problem (2.12)-(2.13)

4.1 Preliminaries. Notion of "weak approximate solution"

We start by considering an other grid $\mathcal{L}$ associated with the primary grid (see Figure 6 below). The elements of $\mathcal{L}$ are made of open rhombi L completely imbedded in $\overline{\Omega}$. We denote Γ_L the boundary of $L \in \mathcal{L}$ and $\mathbf{E}(\mathcal{L})$ the space of functions v defined almost everywhere in Ω such that v is constant in every $L \in \mathcal{L}$ and zero elsewhere. This space is obviously non-empty since there is the null function.

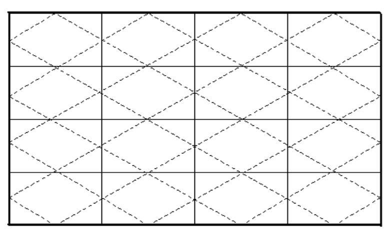

Figure 6: An example of grid $\mathcal{L}$ made up of rhombi associated with a primary rectangular grid.

Let us endow $\mathbf{E}(\mathcal{L})$ with the following discrete energy norm. For all $v \in \mathbf{E}(\mathcal{L})$ we set:

$$\|v\|_{1,h} = \left[\sum_{s \in S} (\Delta_s v)\right]^{\frac{1}{2}} \tag{4.1}$$

where

$$(\Delta_s v) = \sum_{\substack{L,K \in \mathcal{L} \text{ such that} \\ \Gamma_K \cap \Gamma_L = \{s\}}} |v_L - v_K|^2 \tag{4.2}$$

and where S is the set of vertices associated with the grid $\mathcal{L}$.

Note that a vertex $s \in S$ could belong to the boundary Γ of the domain Ω. In this case there exists a unique element L of $\mathcal{L}$ such that s belongs to the boundary Γ_L of L. It is therefore natural to define $(\Delta_s v)$ in this case by $(\Delta_s v) = |v_L|^2$. The norm defined by (4.1) could be viewed as a discrete version of the classical $H^1_0(\Omega)$ norm.

Let us introduce the space

$$C_0\left(\overline{\Omega}\right) = \{v : \overline{\Omega} \longrightarrow \mathbb{R} \text{ is continuous, and } v = 0 \text{ on } \Gamma\}$$

and the following operator :

$$\Pi : \quad C_0\left(\overline{\Omega}\right) \longrightarrow E\left(\mathcal{L}\right)$$

$$v \mapsto \Pi v$$

with:

$$[\Pi v]\,(x) = \begin{cases} v(x_{_L}), \text{ if } x \in \ Int(L), \text{ with } L \in \mathcal{L} \\ \\ 0 \text{ if } \ x \in \ \overline{\Omega} \setminus \left(\bigcup\limits_{L\in\mathcal{L}} L\right) \end{cases} \tag{4.3}$$

where $L \in \mathcal{L}$ and where $x_{_L} = (x_{_1}^L, x_{_2}^L)^t$ are the coordinates of the center of L.

Since the approximate solutions $U_{\mathcal{T}}^h, U_{\mathcal{S}}^h, U_{\mathcal{P}}^h$ of the diffusion problem(1.1)-(1.2) are in $C_0\left(\overline{\Omega}\right)$ (see Propositions 3.2, 3.4 and 3.6), $\Pi U_{\mathcal{T}}^h$, $\Pi U_{\mathcal{S}}^h$ and $\Pi U_{\mathcal{P}}^h$ exist and are unique.

Definition 4.1 Let v be a function of $\mathbf{E}(\mathcal{L})$. v is a weak approximate solution for the diffusion problem (1.1)-(1.2) if there exists an approximate solution V of (1.1)-(1.2) in the sense of either Definition 3.1, Definition 3.3 or Definition 3.5 such that $v = \Pi V$. $\Diamond$

remark 4.2 According to this definition, $\Pi U_{\mathcal{T}}^h$, $\Pi U_{\mathcal{S}}^h$ and $\Pi U_{\mathcal{P}}^h$ are weak approximate solutions of (1.1)-(1.2). Moreover they define the same weak approximate solution denoted u_h in the sequel, for the sake of simplicity of notations and clarity of the presentation. $\Diamond$

4.2 Stability of the weak approximate solution

We are going to prove here the stability of the weak approximate solution in the sense of the discrete energy norm (4.1). The main ingredient for the proof of this result is a discrete version of the Poincaré inequality which reads as follows.

Lemma 4.3 (discrete version of Poincaré inequality)

There exists a strictly positive number P, independent of h, such that

$$\|v\|_{L^2(\Omega)} \leq P\,\|v\|_{1,h} \qquad\qquad \forall\ v \in \mathbf{E}(\mathcal{L})$$

where we have set

$$\|v\|_{L^2(\Omega)} = \left(\int_\Omega v^2 dx\right)^{\frac{1}{2}} \quad \Diamond$$

Proof. See [NN 06]. ■

Proposition 4.4 (Stability result)
The weak approximate solution u_h of the diffusion problem (1.1)-(1.2) satisfies the following inequality:

$$\|u_h\|_{1,h} \leq C \|f\|_{L^2(\Omega)}$$

where C is a strictly positive real number not depending on the spatial discretization ♢

Proof. Multiplying (2.12) and (2.13) by $u_{i,j}$ and $u_{i+\frac{1}{2},j+\frac{1}{2}}$ respectively, and besides summing for $i,j \in \{1,2,...,N\}$ and for $i,j \in \{1,2,...,N-1\}$ respectively yields (with utilization of notation (2.21) and matrix formulation (2.20)):

$$[U_{cc} \quad U_{vc}] \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} U_{cc} \\ U_{vc} \end{bmatrix} = [U_{cc} \quad U_{vc}] \begin{bmatrix} F_{cc} \\ F_{vc} \end{bmatrix} \tag{4.4}$$

Let us recall that in the previous section, we have set (see the relation (2.20)):

$$F_{cc} = \left\{ \int_{K_{i,j}} f dx \right\}_{1 \leq i,j \leq N} \qquad \text{and} \qquad F_{vc} = \left\{ \int_{K_{i+\frac{1}{2},\ j+\frac{1}{2}}} f dx \right\}_{1 \leq i,j \leq N} \tag{4.5}$$

Let us set also

$$LHS = [U_{cc}\ U_{vc}] \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} U_{cc} \\ U_{vc} \end{bmatrix}$$

We have proven that (see the relation (2.25))

$$\begin{aligned} LHS \geq & \sum_{\substack{1 \leq i \leq N \\ 0 \leq j \leq N}} \lambda_{\min}^{ij,ij+1} \left\{ \left(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i-\frac{1}{2},j+\frac{1}{2}} \right)^2 + \left(u_{i,j+1} - u_{i,j} \right)^2 \right\} \\ & + \sum_{\substack{0 \leq i \leq N \\ 1 \leq j \leq N}} \lambda_{\min}^{ij,i+1j} \left\{ \left(u_{i+1,j} - u_{i,j} \right)^2 + \left(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i+\frac{1}{2},j-\frac{1}{2}} \right)^2 \right\} \end{aligned} \tag{4.6}$$

Remark that there exists a strictly positive real number γ only depending on the geological structure of the medium and satisfying the following relation:

$$0 < \gamma = \min \left\{ \min_{1 \leq i \leq N,\ 0 \leq j \leq N} \lambda_{\min}^{ij,ij+1},\ \min_{0 \leq i \leq N,\ 1 \leq j \leq N} \lambda_{\min}^{ij,i+1j} \right\}$$

Thus

$$\begin{aligned} LHS \geq \gamma \Bigg[& \sum_{\substack{1 \leq i \leq N \\ 0 \leq j \leq N}} \left\{ \left(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i-\frac{1}{2},j+\frac{1}{2}} \right)^2 + \left(u_{i,j+1} - u_{i,j} \right)^2 \right\} \\ & + \sum_{\substack{0 \leq i \leq N \\ 1 \leq j \leq N}} \left\{ \left(u_{i+1,j} - u_{i,j} \right)^2 + \left(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i+\frac{1}{2},j-\frac{1}{2}} \right)^2 \right\} \Bigg] \end{aligned} \tag{4.7}$$

From the proof of Proposition 2.3 we know that the left hand side of (4.4) satisfies the following inequality :

$$\gamma \left\| u_h \right\|_{1,h}^2 \leq LHS \tag{4.8}$$

In addition, the right hand side of (4.4) obeys to the following relation

$$\left| \begin{bmatrix} U_{cc} & U_{vc} \end{bmatrix} \begin{bmatrix} F_{cc} \\ F_{vc} \end{bmatrix} \right| \leq \sum_{c \in C} \sum_{s \in A(c)} \left[\int_{Kcs} \left(|f| \left| U_s + U_c \right| \right) dx \right] \equiv RHS \tag{4.9}$$

where C is the set of grid-block centers of the primary grid, $A(c)$ the set of vertices of grid-blocks centered on c, with $c \in C$, K_{cs} the quarter of a grid-block from the primary mesh, containing the points c and s, with $c \in C$ and $s \in A(c)$ (see Figure 7 below), u_s value of u_h in the element of $\mathcal{L}$ centered on s, idem for u_c.

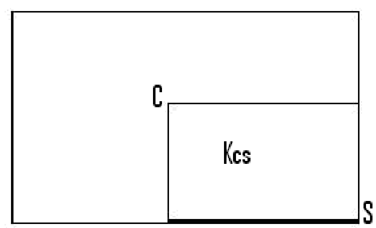

Figure 7: An element of the primary mesh including a sub-element K_{cs}

An application of Cauchy-Schwarz inequality to the integral term of (4.9) yields

$$\sum_{c \in C} \sum_{s \in A(c)} \left[\int_{Kcs} \left(|f| \left| U_s + U_c \right| \right) dx \right] \leq \sum_{c \in C} \sum_{s \in A(c)} \left[\frac{h}{2} \left(|U_s| + |U_c| \right) \left(\int_{K_{cs}} f^2 dx \right)^{\frac{1}{2}} \right]$$

A double application of discrete Cauchy-Schwarz inequality leads to

$$RHS \leq 2 \left[\sum_{c \in C} \sum_{s \in A(c)} \int_{K_{cs}} f^2 dx \right]^{\frac{1}{2}} \left[\sum_{c \in C} \sum_{s \in A(c)} \frac{h^2}{8} \left(U_s^2 + U_c^2 \right) \right]^{\frac{1}{2}} \tag{4.10}$$

Remarking that

$$\sum_{c \in C} \sum_{s \in A(c)} \frac{h^2}{8} \left(U_s^2 + U_c^2 \right) = \int_{\Omega} |u_h|^2 \, dx$$

and thanks to (4.8)-(4.10) together with Lemma 4.3, the proof of Proposition 4.4 is ended ■

REMARK 4.5 This stability result implies the L^2-stability of the weak approximate solution. This follows from Lemma 4.3. ◊

4.3 Error estimates for the weak approximate solution

When accounting with the truncation error, the equations (2.8)-(2.11) are transformed as follows :

$$\begin{aligned}
&\frac{2D_{22}^{ij}D_{22}^{ij+1}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[\varphi_{i,j}-\varphi_{i,j+1}\right]+\frac{D_{22}^{ij}D_{21}^{ij+1}+D_{22}^{ij+1}D_{21}^{ij}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[\varphi_{i-\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\right]\\
&+\frac{2D_{22}^{ij}D_{22}^{ij-1}}{D_{22}^{ij}+D_{22}^{ij-1}}\left[\varphi_{i,j}-\varphi_{i,j-1}\right]+\frac{D_{22}^{ij}D_{12}^{ij-1}+D_{22}^{ij-1}D_{12}^{ij}}{D_{22}^{ij}+D_{22}^{ij-1}}\left[\varphi_{i+\frac{1}{2},j-\frac{1}{2}}-\varphi_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\
&+\frac{2D_{11}^{ij}D_{11}^{i+1j}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[\varphi_{i,j}-\varphi_{i+1,j}\right]+\frac{D_{11}^{ij}D_{21}^{i+1j}+D_{11}^{i+1j}D_{21}^{ij}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[\varphi_{i+\frac{1}{2},j-\frac{1}{2}}-\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\right]\\
&+\frac{2D_{11}^{ij}D_{11}^{i-1j}}{D_{11}^{ij}+D_{11}^{i-1j}}\left[\varphi_{i,j}-\varphi_{i-1,j}\right]+\frac{D_{11}^{ij}D_{12}^{i-1j}+D_{11}^{i-1j}D_{12}^{ij}}{D_{11}^{ij}+D_{11}^{i-1j}}\left[\varphi_{i-\frac{1}{2},j+\frac{1}{2}}-\varphi_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\
&=\int_{K_{ij}}f(x)dx\quad+\sum_{e\in E_{i,j}}hR_{i,j}^{e}\qquad\forall\quad 1\leq\ i,\ j\leq\ N
\end{aligned}\tag{4.11}$$

$$\begin{aligned}
&\frac{D_{11}^{ij+1}D_{21}^{i+1j+1}+D_{11}^{i+1j+1}D_{21}^{ij+1}}{D_{11}^{ij+1}+D_{11}^{i+1j+1}}\left[\varphi_{i,j+1}-\varphi_{i+1,j+1}\right]+\\
&\left(\frac{\left(D_{12}^{i+1j+1}-D_{12}^{ij+1}\right)\left(D_{21}^{ij+1}-D_{21}^{i+1j+1}\right)}{2\left(D_{11}^{ij+1}+D_{11}^{i+1j+1}\right)}+\frac{D_{22}^{ij+1}+D_{22}^{i+1j+1}}{2}\right)\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j+\frac{3}{2}}\right]\\
&+\frac{D_{11}^{i+1j}D_{21}^{ij}+D_{11}^{ij}D_{21}^{i+1j}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[\varphi_{i+1,j}-\varphi_{i,j}\right]+\\
&\left(\frac{\left(D_{12}^{i+1j}-D_{12}^{ij}\right)\left(D_{21}^{ij}-D_{21}^{i+1j}\right)}{2\left(D_{11}^{ij}+D_{11}^{i+1j}\right)}+\frac{D_{22}^{ij}+D_{22}^{i+1j}}{2}\right)\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j-\frac{1}{2}}\right]\\
&+\frac{D_{22}^{i+1j}D_{12}^{i+1j+1}+D_{22}^{i+1j+1}D_{12}^{i+1j}}{D_{22}^{i+1j}+D_{22}^{i+1j+1}}\left[\varphi_{i+1,j}-\varphi_{i+1,j+1}\right]+\\
&\left(\frac{\left(D_{21}^{i+1j+1}-D_{21}^{i+1j}\right)\left(D_{12}^{i+1j}-D_{12}^{i+1j+1}\right)}{2\left(D_{22}^{i+1j}+D_{22}^{i+1j+1}\right)}+\frac{D_{11}^{i+1j}+D_{11}^{i+1j+1}}{2}\right)\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{3}{2},j+\frac{1}{2}}\right]\\
&+\frac{D_{22}^{ij+1}D_{12}^{ij}+D_{22}^{ij}D_{12}^{ij+1}}{D_{22}^{i+1j}+D_{22}^{i+1j+1}}\left[\varphi_{i,j+1}-\varphi_{i,j}\right]+\\
&\left(\frac{\left(D_{21}^{ij+1}-D_{21}^{ij}\right)\left(D_{12}^{ij}-D_{12}^{ij+1}\right)}{2\left(D_{22}^{ij}+D_{22}^{ij+1}\right)}+\frac{D_{11}^{ij}+D_{11}^{ij+1}}{2}\right)\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i-\frac{1}{2},j+\frac{1}{2}}\right]\\
&=\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}f(x)dx\quad+\sum_{e\in E_{i+\frac{1}{2},j+\frac{1}{2}}}hR_{i+\frac{1}{2},j+\frac{1}{2}}^{e}\quad\forall\quad 1\leq\ i,\ j\leq\ N\ -1
\end{aligned}\tag{4.12}$$

with the following discrete boundary conditions :

$$\varphi_{i+\frac{1}{2},\frac{1}{2}}=\varphi_{i+\frac{1}{2},N+\frac{1}{2}}=\varphi_{\frac{1}{2},j+\frac{1}{2}}=\varphi_{N+\frac{1}{2},j+\frac{1}{2}}=0\qquad\forall\quad 0\leq\ i,\ j\leq\ N\tag{4.13}$$

$$\varphi_{i,0} = \varphi_{0,j} = \varphi_{i,N+1} = \varphi_{N+1,j} = 0 \qquad \forall \quad 1 \leq \; i, \; j \leq \; N \tag{4.14}$$

where $E_{i,j}$ and $E_{i+\frac{1}{2},j+\frac{1}{2}}$ are sets of edges associated respectively with $K_{i,j}$ and $K_{i+\frac{1}{2},j+\frac{1}{2}}$, and where $R^e_{i,j}$ and $R^e_{i+\frac{1}{2},j+\frac{1}{2}}$ denote the truncation error associated with the approximation of the flux over the edges $e_{i,j}$ and $e_{i+\frac{1}{2},j+\frac{1}{2}}$ respectively. Moreover, under the assumption $\varphi \in C^2$ over the closure of primary grid-blocks, the truncation error satisfy the following inequalities :

$$\left| R^e_{i,j} \right| \leq Ch \qquad \text{and} \qquad \left| R^e_{i+\frac{1}{2},j+\frac{1}{2}} \right| \leq Ch \tag{4.15}$$

In what follows, the notation R^e_K will be used to denote the truncation error for the approximation of the flux over the edge e_K of any control volume K. Due to the conservativity property of the proposed finite volume formulation, we have

$$R^e_K \quad + \quad R^e_I = 0 \tag{4.16}$$

where K and I are two adjacent control volumes such that $e = \Gamma_K \cap \Gamma_I$.
Let us define a function ε_h almost everywhere in $\mathbb{R}^2$ in the following way :

$$\varepsilon_h\,(x) = \begin{cases} \varepsilon_L & \text{if } \; x \in Int(L) \\ \quad 0 & \text{elsewhere} \end{cases} \qquad \text{with} \quad L \in \mathcal{L} \tag{4.17}$$

where we have set $\varepsilon_L = \varphi_L - u_L$ for all $L \in \mathcal{L}$. Note that the element L of the mesh $\mathcal{L}$ is necessary centered on a point whose cartesian coordinates are of the form $\left(x_1^i, x_2^j\right)$ or $\left(x_1^{i+\frac{1}{2}}, x_2^{j+\frac{1}{2}}\right)$. ε_L is a generic name of $\varepsilon_{i,j}$ or $\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}$.

REMARK 4.6 From the relation (4.16) we see that the function ε_h is actually in the space $\mathbf{E}(\mathcal{L})$. This function expresses the error in some sense (i.e. the difference between the exact and the weak approximate solution u_h) and certain estimates of this error are given in what follows. $\Diamond$

We immediately are going to show that the following quantities $\{\varepsilon_{i,\,j}\}_{1\leq\, i,\; j\leq\; N}$ and $\left\{\varepsilon_{i+\frac{1}{2},\,j+\frac{1}{2}}\right\}_{1\leq\, i,\; j\leq\; N-1}$ are a solution of a discrete problem of the form (2.12)-(2.15). Subtracting (2.12) from (4.11) and (2.13) from (4.12), and reordering the terms yields :

$$\begin{aligned}
&\frac{2D_{22}^{ij}D_{22}^{ij+1}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[\varepsilon_{i,j}-\varepsilon_{i,j+1}\right]+\frac{D_{22}^{ij}D_{21}^{ij+1}+D_{22}^{ij+1}D_{21}^{ij}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[\varepsilon_{i-\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}\right]\\
&+\frac{2D_{22}^{ij}D_{22}^{ij-1}}{D_{22}^{ij}+D_{22}^{ij-1}}\left[\varepsilon_{i,j}-\varepsilon_{i,j-1}\right]+\frac{D_{22}^{ij}D_{12}^{ij-1}+D_{22}^{ij-1}D_{12}^{ij}}{D_{22}^{ij}+D_{22}^{ij-1}}\left[\varepsilon_{i+\frac{1}{2},j-\frac{1}{2}}-\varepsilon_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\
&+\frac{2D_{11}^{ij}D_{11}^{i+1j}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[\varepsilon_{i,j}-\varepsilon_{i+1,j}\right]+\frac{D_{11}^{ij}D_{21}^{i+1j}+D_{11}^{i+1j}D_{21}^{ij}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[\varepsilon_{i+\frac{1}{2},j-\frac{1}{2}}-\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}\right]\\
&+\frac{2D_{11}^{ij}D_{11}^{i-1j}}{D_{11}^{ij}+D_{11}^{i-1j}}\left[\varepsilon_{i,j}-\varepsilon_{i-1,j}\right]+\frac{D_{11}^{ij}D_{12}^{i-1j}+D_{11}^{i-1j}D_{12}^{ij}}{D_{11}^{ij}+D_{11}^{i-1j}}\left[\varepsilon_{i-\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\
&=\sum_{e\in E_{i,j}} hR^e_{i,j} \qquad \forall \quad 1\leq\; i,\; j\leq\; N
\end{aligned} \tag{4.18}$$

$$\frac{D_{11}^{ij+1}D_{21}^{i+1j+1}+D_{11}^{i+1j+1}D_{21}^{ij+1}}{D_{11}^{ij+1}+D_{11}^{i+1j+1}}\left[\varepsilon_{i,j+1}-\varepsilon_{i+1,j+1}\right] \quad +$$

$$\left(\frac{\left(D_{12}^{i+1j+1}-D_{12}^{ij+1}\right)\left(D_{21}^{ij+1}-D_{21}^{i+1j+1}\right)}{2\left(D_{11}^{ij+1}+D_{11}^{i+1j+1}\right)}+\frac{D_{22}^{ij+1}+D_{22}^{i+1j+1}}{2}\right)\left[\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i+\frac{1}{2},j+\frac{3}{2}}\right] \quad +$$

$$\frac{D_{11}^{i+1j}D_{21}^{ij}+D_{11}^{ij}D_{21}^{i+1j}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[\varepsilon_{i+1,j}-\varepsilon_{i,j}\right] \quad +$$

$$\left(\frac{\left(D_{12}^{i+1j}-D_{12}^{ij}\right)\left(D_{21}^{ij}-D_{21}^{i+1j}\right)}{2\left(D_{11}^{ij}+D_{11}^{i+1j}\right)}+\frac{D_{22}^{ij}+D_{22}^{i+1j}}{2}\right)\left[\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i+\frac{1}{2},j-\frac{1}{2}}\right] \quad +$$

$$\frac{D_{22}^{i+1j}D_{12}^{i+1j+1}+D_{22}^{i+1j+1}D_{12}^{i+1j}}{D_{22}^{i+1j}+D_{22}^{i+1j+1}}\left[\varepsilon_{i+1,j}-\varepsilon_{i+1,j+1}\right] \quad + \tag{4.19}$$

$$\left(\frac{\left(D_{21}^{i+1j+1}-D_{21}^{i+1j}\right)\left(D_{12}^{i+1j}-D_{12}^{i+1j+1}\right)}{2\left(D_{22}^{i+1j}+D_{22}^{i+1j+1}\right)}+\frac{D_{11}^{i+1j}+D_{11}^{i+1j+1}}{2}\right)\left[\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i+\frac{3}{2},j+\frac{1}{2}}\right]$$

$$\frac{D_{22}^{ij+1}D_{12}^{ij}+D_{22}^{ij}D_{12}^{ij+1}}{D_{22}^{i+1j}+D_{22}^{i+1j+1}}\left[\varepsilon_{i,j+1}-\varepsilon_{i,j}\right] \quad +$$

$$\left(\frac{\left(D_{21}^{ij+1}-D_{21}^{ij}\right)\left(D_{12}^{ij}-D_{12}^{ij+1}\right)}{2\left(D_{22}^{ij}+D_{22}^{ij+1}\right)}+\frac{D_{11}^{ij}+D_{11}^{ij+1}}{2}\right)\left[\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i-\frac{1}{2},j+\frac{1}{2}}\right]$$

$$=\sum_{e\in E_{i+\frac{1}{2},j+\frac{1}{2}}} hR^e_{i+\frac{1}{2},j+\frac{1}{2}} \qquad \forall\, 1\le i,\ j\le N-1$$

where, thanks to the discrete boundary conditions (2.14), (2.15), (4.13) and (4.14), we have naturally:

$$\varepsilon_{i+\frac{1}{2},\frac{1}{2}}=\varepsilon_{i+\frac{1}{2},N+\frac{1}{2}}=\varepsilon_{\frac{1}{2},j+\frac{1}{2}}=\varepsilon_{N+\frac{1}{2},j+\frac{1}{2}}=0 \qquad \forall \quad 0\le\ i,\ j\le\ N \tag{4.20}$$

$$\varepsilon_{i,0}=\varepsilon_{0,j}=\varepsilon_{i,N+1}=\varepsilon_{N+1,j}=0 \qquad \forall \quad 1\le\ i,\ j\le\ N \tag{4.21}$$

Multiplying (4.18) and (4.19) by $\varepsilon_{i,j}$ and $\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}$ respectively, summing over i,j and reordering the terms of the left hand side after summation side by side of the two final equations, leads to the following inequality, accounting with (1.3):

$$\begin{aligned}
\gamma\,\|\varepsilon_h\|_{1,h}^2 &\leq \sum_{1\leq i,\ j\leq N}\left(h\ \varepsilon_{i,j}\sum_{e\in E_{i,j}} R_{i,j}^e\right)\\
&+ \sum_{1\leq i,\ j\leq N-1} h\ \varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}\left(\sum_{e\in E_{i+\frac{1}{2},j+\frac{1}{2}}} R^e_{i+\frac{1}{2},j+\frac{1}{2}}\right)\\
&\leq h\sum_{1\leq i\leq N,\ 1\leq j\leq N} a_{i,j}\left[\left|\varepsilon_{i,j}-\varepsilon_{i,j+1}\right| + \left|\varepsilon_{i-\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}\right|\right]\\
&+ h\sum_{0\leq i\leq N,\ 1\leq j\leq N} b_{i,j}\left[\left|\varepsilon_{i,j}-\varepsilon_{i+1,j}\right| + \left|\varepsilon_{i-\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}\right|\right]\\
&\leq \text{(double application of Cauchy-Schwarz inequality)}\\
&\leq 2h\left[\sum_{1\leq i\leq N,\ 0\leq j\leq N} a_{i,j}^2 + \sum_{0\leq i\leq N,\ 1\leq j\leq N} b_{i,j}^2\right]^{\frac{1}{2}}\|\varepsilon_h\|_{1,h}
\end{aligned}$$

where we have set, for $1\leq i\leq N$ and $0\leq j\leq N$:

$$a_{i,j}=\max\left\{R_{i,j}\ ,\ R_{i-\frac{1}{2},j+\frac{1}{2}}\right\}$$

with

$$R_{i,j}=\max_e\left|R_{i,j}^e\right|,\ R_{i-\frac{1}{2},j+\frac{1}{2}}=\max_e\left|R^e_{i-\frac{1}{2},j+\frac{1}{2}}\right|$$

and for $0\leq i\leq N$ and $1\leq j\leq N$:

$$b_{i,j}=\max\left\{R_{i,j}\ ,\ R_{i+\frac{1}{2},j-\frac{1}{2}}\right\}$$

with

$$R_{i,j}=\max_e\left|R_{i,j}^e\right|,\ R_{i+\frac{1}{2},j-\frac{1}{2}}=\max_e\left|R^e_{i+\frac{1}{2},j-\frac{1}{2}}\right|$$

Note that, due to (4.15), the two summations in the previous inequality are bounded by a constant independent of h. Therefore, we deduce thanks to (4.15) that if $\varphi\in C^2\left(\overline{K}\right)$ for any grid-block K, we have

$$\|\varepsilon_h\|_{1,h}\leq Ch \tag{4.22}$$

where C is a positive real number depending exclusively on φ, Ω and γ.

Let us now investigate the error estimate for the $L^\infty-norm$ defined over the space

$\mathbf{E}(\mathcal{L})$ by:

$$\|v_h\|_{L^\infty(\Omega)}=\max_L|v_L| \quad \text{or, which is equivalent, } \|v_h\|_{L^\infty(\Omega)}=\max_{1\leq i,j\leq N}|v_{i,j}|\ .$$

Since $\varepsilon_{0,j}=0$ (see relation (4.21)), it is obvious that

$$\varepsilon_{i,j}=-\varepsilon_{0,j}+\varepsilon_{1,j}\ -\varepsilon_{1,j}+\varepsilon_{2,j}-\varepsilon_{2,j}+...+\varepsilon_{i-1,j}-\varepsilon_{i-1,j}+\varepsilon_{i,j}\qquad\forall 1\leq i,j\leq N$$

Then, thanks to Minkowski inequality, one deduces that

$$\begin{aligned} |\varepsilon_{i,j}| &\leq |-\varepsilon_{0,j}+\varepsilon_{1,j}| + |-\varepsilon_{1,j}+\varepsilon_{2,j}| + ... + |-\varepsilon_{i-1,j}+\varepsilon_{i,j}| \\ &\leq \sum_{k=0}^{N} |\varepsilon_{k+1,j}-\varepsilon_{k,j}| = \sum_{k=0}^{N} h^{\frac{1}{2}} \frac{|\varepsilon_{k+1,j}-\varepsilon_{k,j}|}{h^{\frac{1}{2}}} \qquad \forall\, 1 \leq i,j \leq N \end{aligned}$$

with $\varepsilon_{N+1,j} = 0$ (see relation (4.21)). Applying Cauchy-Schwarz inequality leads to:

$$\max_{1\leq i,j\leq N} |\varepsilon_{i,j}| \leq \sqrt{2} h^{-\frac{1}{2}} \left\|\varepsilon_h\right\|_{1,h}$$

It then follows, utilizing (4.22), that

$$\|\varepsilon_h\|_{L^\infty(\Omega)} \leq \sqrt{2} C h^{\frac{1}{2}} \tag{4.23}$$

Let us summarize these error estimates (i.e. (4.22) and (4.23)) in the following assertion.

Theorem 4.7 (Error estimates in following norms: $\mathbf{L}^\infty(\Omega)$ and $\|.\|_{1,h}$)
Assume that the diffusion tensor D in the Diffusion problem (1.1)-(1.2) is a full matrix which is symmetric and positive definite, with piecewise constant coefficients. Assume also that the unique variational solution φ of (1.1)-(1.2) satisfies $\varphi_{|\overline{K}} \in C^2\left(\overline{K}\right)$, for any grid-block K of the primary grid. Let us consider the space $\mathbf{E}(\mathcal{L})$ made up of functions v defined almost everywhere in Ω such that v is constant in each rhombus of the mesh $\mathcal{L}$ and zero elsewhere (see Figure 6 for the definition of $\mathcal{L}$). Let us recall that $u_h = \Pi U^h_{\mathcal{T}}$ and set $\varphi_h = \Pi\varphi$, where Π is an operator introduced in sub-section 4.1.

Then, the function $\varepsilon_h = \varphi_h - u_h$ satisfies the following inequalities:

$$\begin{aligned} (i) \qquad & \|\varepsilon_h\|_{1,h} \leq C\,h \\ (ii) \qquad & \|\varepsilon_h\|_{L^\infty(\Omega)} \leq \sqrt{2}\,C\,h^{\frac{1}{2}} \end{aligned}$$

where C is a strictly positive real number depending exclusively on φ, Ω and γ. ◊

Corollary 4.8 (Error estimate in $L^2(\Omega) - norm$)
ε_h satisfies the following inequality

$$\|\varepsilon_h\|_{L^2(\Omega)} \leq P\,h\cdot$$

where P is a strictly positive constant independent of h. ◊

Proof. This inequality follows obviously from the inequality (i) of Theorem 4.7 and Lemma 4.3. ■

remark 4.9 Note that in the earlier work of [NN 06], error estimates of the weak approximate solution of the boundary-value problem (1.1)-(1.2) has been investigated in the case of anisotropic homogeneous porous media. In this framework the

following results have been proven:
If the exact solution φ of (1.1)-(1.2) is in $C^3(\overline{\Omega})$ then

$$(i) \qquad \|\varepsilon_h\|_{1,h} \leq C\, h^2$$

$$(ii) \qquad \|\varepsilon_h\|_{L^\infty(\Omega)} \leq C\, h^{\frac{3}{2}}$$

$$(iii) \qquad \|\varepsilon_h\|_{L^2(\Omega)} \leq C\, h^2$$

where C represents miscellaneous strictly positive constants without dependence on h. $\Diamond$

5 Convergence results

The following classical results (see for instance [RT 83], page 103) from the interpolation theory will be needed. In what follows, C denotes miscellaneous constants without dependence on h and Λ is the classical Lagrange interpolation operator associated respectively with the nodes of the grid $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{P}$.

Lemma 5.1 Let $(K, P(K), \sum)$ be a finite element of type P_1 or Q_1. Then there exists a constant C not depending on the grid size h, such that for $m \in \{0,1,2\}$, the following estimate holds:

$$\forall v \in H^2(K), \qquad \left(\sum_{|\alpha|=m} \|D^\alpha (v - \Lambda v)\|^2_{L^2(K)} \right)^{\frac{1}{2}} \leq C\, h^{2-m} \left(\sum_{|\alpha|=2} \|D^\alpha v\|^2_{L^2(K)} \right)^{\frac{1}{2}}. \quad \Diamond$$

Lemma 5.2 Let $(E, P(E), \sum)$ be a finite element of type Q_2. Then there exists a constant C not depending on the grid size h, such that for $m \in \{0,1,2,3\}$, the following estimate holds:

$$\forall v \in H^3(E), \qquad \left(\sum_{|\alpha|=m} \|D^\alpha (v - \Lambda v)\|^2_{L^2(E)} \right)^{\frac{1}{2}} \leq C\, h^{3-m} \left(\sum_{|\alpha|=3} \|D^\alpha v\|^2_{L^2(E)} \right)^{\frac{1}{2}}. \quad \Diamond$$

We are going to carry out the error estimates for diverse approximate solutions introduced in the section 3. The notations introduced in the previous sections are conserved here. In this connection, we recall that φ denotes the exact solution of the boundary-value problem (1.1)-(1.2). On the other hand, $U^h_{\mathcal{T}}$, $U^h_{\mathcal{S}}$ and $U^h_{\mathcal{P}}$ denote respectively the linear, bilinear and biquadratic approximate solutions of (1.1)-(1.2).

5.1 Error estimates for the piecewise linear approximate solution $U^h_{\mathcal{T}}$

We investigate in this subsection the error estimates for the piecewise linear approximate solution introduced in section 3. In what follows, C represents miscellaneous strictly positive constants without dependence on h.

Let T be the generic name of triangular elements from $\mathcal{T}$ and let us now deal with the estimates of $\left\|\Lambda\varphi - U^h_{\mathcal{T}}\right\|_{0,\Omega}$. One may set that

$$U^h_{\mathcal{T}}(x) = \sum_{n\in N} U_n \Phi_n(x) \qquad \textit{and} \qquad \Lambda\varphi(x) = \sum_{n\in N} \varphi_n \Phi_n(x)$$

where N denotes the set of nodes associated with $\mathcal{T}$, Φ_n is the generic name of classical basis functions from $\mathcal{P}_1$ finite element theory, U_n and φ_n are respectively the value of $U^h_{\mathcal{T}}$ and φ at the node number n and where Λ denotes the classical Lagrange operator of interpolation associated with the primary grid-block centers and corners.

Combining the two preceding relations leads to

$$(\Lambda\varphi - U^h_{\mathcal{T}})(x) = \sum_{n\in N} \varepsilon_n \Phi_n(x) \tag{5.1}$$

where we have set $\varepsilon_n = \varphi(x^n) - U^h_{\mathcal{T}}(x^n)$, with x^n representing the coordinates of the node number n from the grid $\mathcal{T}$.

Let us investigate an estimate of the error $\Lambda\varphi - U^h_{\mathcal{T}}$ in $L^2(\Omega)-$norm. In this connection, the following well-known relations will play a key role.

$$0 \leq \Phi_n(x) \leq 1 \quad \forall\, x \in \overline{\Omega} \qquad \text{and} \qquad \sum_{n\in N} \Phi_n(x) = 1$$

From Cauchy-Schawrz inequality we get

$$|(\wedge\varphi - U^h_T)(x)| = |\sum_n \varepsilon_n \Phi_n(x)| \leq \left(\sum_n \varepsilon_n^2 \Phi_n(x)\right)^{\frac{1}{2}} \left(\sum_n \Phi_n(x)\right)^{\frac{1}{2}}$$

Since

$$\sum_n \Phi_n(x) = 1$$

we have

$$\begin{aligned}
\int_\Omega |(\wedge\varphi - U^h_T)(x)|^2 dx &\leq \int_\Omega \left(\sum_n \varepsilon_n^2 \Phi_n(x)\right) dx \\
&= \sum_n \varepsilon_n^2 \left(\int_\Omega \Phi_n(x)\right) dx \\
&= \sum_n \varepsilon_n^2 \left(\int_{supp(\Phi_n)} \Phi_n(x)\right) dx \\
&= \sum_n \frac{2}{3} mes(L_n)\varepsilon_n^2 \\
&= \frac{2}{3} \sum_n \int_{L_n} |\varepsilon_h(x)|^2\, dx \\
&= \frac{2}{3} \left\|\varepsilon_h\right\|^2_{L^2(\Omega)}
\end{aligned}$$

where $mes(.)$ denotes the Lebesgue measure in $\mathbb{R}^2$, and where L_n is a rhombus of $\mathcal{L}$ centered on the node n. Thanks to Theorem 4.7 and Remark 4.9 we deduce that

$$\|\wedge\varphi - U_T^h\|_{L^2(\Omega)} \leq \begin{cases} C\,h^2 & \text{if the permeability tensor } D \text{ is uniformly constant} \\ C\,h & \text{if the permeability tensor } D \text{ is piecewise constant} \end{cases}$$

We are concerned now with the estimates of $(\wedge\varphi - U_T^h)$ in $L^\infty(\Omega)$. It is clear that

$$\left|(\Lambda\varphi - U_T^h)(x)\right| \leq \max_{n\in N} |\varepsilon_n| \sum_{n\in N} \Phi_n(x) \qquad in \quad \Omega$$

It follows from Theorem 4.7 and Remark 4.9 that for $x \in \Omega$ we have:

$$\left|(\Lambda\varphi - U_T^h)(x)\right| \leq \begin{cases} C\,h^{\frac{3}{2}} & \text{if the permeability tensor } D \text{ is uniformly constant} \\ C\,h^{\frac{1}{2}} & \text{if the permeability tensor } D \text{ is piecewise constant} \end{cases}$$

Let us summarize the preceding estimates as it follows.

Proposition 5.3 Under the same assumptions as those of Theorem 4.7, the exact solution φ and the piecewise linear approximate solution U_T^h satisfy the following estimates:

$$\| \wedge \varphi - U_T^h\|_{L^2(\Omega)} \leq C\,h \tag{5.2}$$

$$\| \wedge \varphi - U_T^h\|_{L^\infty(\Omega)} \leq C\,h^{\frac{1}{2}} \tag{5.3}$$

Moreover, if D is uniformly constant in Ω and $\varphi \in C^3(\overline{\Omega})$ we have:

$$\| \wedge \varphi - U_T^h\|_{L^2(\Omega)} \leq C\,h^2 \tag{5.4}$$

$$\| \wedge \varphi - U_T^h\|_{L^\infty(\Omega)} \leq C\,h^{\frac{3}{2}}. \quad \diamondsuit \tag{5.5}$$

Our aim now is to give the last important results of this subsection in two cases.

• First case: The permeability tensor D is uniform in Ω (which physically means that Ω is homogeneous).

We deduce from Lemma 5.1 (with $m = 0$) and Proposition 5.3 (see inequality (5.4)) the following result.

Proposition 5.4 Assume that the porous medium Ω is homogeneous and that the exact solution φ is in $C^3(\overline{\Omega})$. Then φ and the piecewise linear approximate solution U_T^h satisfy the following estimate:

$$\left\|\varphi - U_T^h\right\|_{0,\Omega} \leq C\,h^2. \quad \diamondsuit$$

• Second case: The permeability tensor D is piecewise constant in Ω (which physically means that Ω is heterogeneous).
Thanks to the Lemma 5.1, we have for $m = 0$

$$\|\varphi - \Lambda\varphi\|_{0,\Omega} \le C\,h^2 \left(\sum_{T\in\mathcal{T}} \|\varphi\|_{2,T}^2 \right)^{\frac{1}{2}}$$

which is equivalent to

$$\|\varphi - \Lambda\varphi\|_{0,\Omega} \le C\,h^2 \left(\sum_{E\in\ \mathcal{P}} \|\varphi\|_{2,E}^2 \right)^{\frac{1}{2}}$$

where $\mathcal{P}$ denotes the primary grid and E the generic name of primary grid-blocks.

Remarking that the summation in the right hand side of the preceding inequality depends actually on the geologic structure rather than the current grid, one can write

$$\|\varphi - \Lambda\varphi\|_{0,\Omega} \le C\,h^2 \left(\sum_{G\in\mathcal{G}} \|\varphi\|_{2,G}^2 \right)^{\frac{1}{2}}$$

where $\mathcal{G}$ denotes the set of geologic formations the medium Ω is made up of. Therefore,

$$\|\varphi - \Lambda\varphi\|_{0,\Omega} \le Ch^2 \tag{5.6}$$

Recall that C denotes miscellaneous constants without dependence on h. This inequality and Proposition 5.3 lead to the following result.

Proposition 5.5 Under the same assumptions as those of Theorem 4.7, the exact solution φ and the piecewise linear approximate solution $U^h_{\mathcal{T}}$ satisfy the following estimate:

$$\left\|\varphi - U^h_{\mathcal{T}}\right\|_{0,\Omega} \le C\,h. \quad \Diamond$$

5.2 Error estimates for the piecewise bilinear and the piecewise biquadratic approximate solutions $U^h_{\mathcal{S}}$ and $U^h_{\mathcal{P}}$

5.2.1 Preliminaries

The discrete energy norm $\|\cdot\|_{1,h}$ introduced in section 4 plays a key role in the error analysis of the linear approximate solution $U^h_{\mathcal{T}}$ (see the derivation of the error estimates from Theorem 4.7 and Corollary 4.8). Unfortunately this discrete energy norm is not suitable for the error analysis of the bilinear and the biquadratic approximate solutions $U^h_{\mathcal{S}}$ and $U^h_{\mathcal{P}}$ since the edge mid-point approximate pressures are involved as degrees of freedom. It is the reason why we need to introduce a suitable discrete energy norm.

First of all, let us recall that a node s is a grid-block center or a grid-block corner or an edge mid-point (with respect to the primary grid). In what follows, $\mathcal{N}$ is the set of the nodes. For s, $p \in \mathcal{N}$, $d(s,p)$ represents the euclidian distance between s and p.

Following the ideas developed in the previous sections, it is natural to introduce a notion of weak-star approximate solution and to investigate its stability and estimates of the associated error in an adequate energy norm, $\|\cdot\|_{L^2(\Omega)}$ −norm and $\|\cdot\|_{L^\infty(\Omega)}$ −norm.

We start by considering a grid $\mathcal{M}$ made up of open uniform square elements of size $\frac{h}{2}$, centered on the nodes $n \in \mathcal{N}$ and completely imbedded in $\overline{\Omega}$. In what follows, M is the generic name of those square elements. We denote $\mathbf{E}(\mathcal{M})$ the space of functions v defined almost everywhere in Ω such that v is constant in every $M \in \mathcal{M}$ and zero elsewhere. This space is obviously non-empty (since there is the null function) and we endow it with the discrete energy norm defined by

$$\|v\|_{2,h} = \left[\sum_{p\,,\,q \in \mathcal{N},\ d(p,q)=\frac{h}{2}} |v_p \, - \, v_q|^2 \right]^{\frac{1}{2}} \tag{5.7}$$

where v_p and v_q represent constant values of v in the square elements from $\mathcal{M}$ centered respectively at nodes p and q.

Let us introduce the following operator

$$\tilde{\Pi}: \quad C_0\left(\overline{\Omega}\right) \longrightarrow E\left(\mathcal{M}\right)$$

$$v \mapsto \tilde{\Pi} v$$

with:

$$\left[\tilde{\Pi} v\right](x) = \begin{cases} v(x_{_M}),\ \text{if } x \in Int(M),\ \text{with } M \in \mathcal{M} \\ \\ \quad 0 \quad \text{if} \quad x \in \overline{\Omega} \setminus \left(\bigcup\limits_{M \in \mathcal{M}} M \right) \end{cases} \tag{5.8}$$

where $M \in \mathcal{M}$, $x_{_M} = (x_1^M, x_2^M)^t$ are the coordinates of the center of M, and where $C_0\left(\overline{\Omega}\right)$ is a space of functions introduced in section 4. Note that $\tilde{\Pi} U_{\mathcal{S}}^h$ and $\tilde{\Pi} U_{\mathcal{P}}^h$ exist and are unique.

Definition 5.6 Let v be a function from $\mathbf{E}(\mathcal{M})$. v is a weak-star approximate solution for the pure diffusion problem (1.1)-(1.2) if there exists an approximate solution V of (1.1)-(1.2) in the sense of either Definition 3.3 or Definition 3.5 such that $v = \tilde{\Pi} V$. $\Diamond$

remark 5.7 According to this definition, $\tilde{\Pi} U_{\mathcal{S}}^h$ and $\tilde{\Pi} U_{\mathcal{P}}^h$ are weak-star approximate solutions of (1.1)-(1.2). Moreover they define the same weak-star approximate solution, denoted $\tilde{u}_h$ in the sequel for the sake of simplicity of notations and clarity of the presentation. $\Diamond$

5.2.2 Stability of the weak-star approximate solution

Let us focus on investigating the stability of the weak-star approximate solution in the sense of the discrete energy norm $\|\cdot\|_{2,h}$. The main ingredient for the proof of this result is the following remark.

REMARK 5.8 Consider the linear operator from $\mathbf{E}(\mathcal{M})$ to $\mathbf{E}(\mathcal{L})$ defined by:
$\tilde{v} \longmapsto v$, with $v_\sigma = \tilde{v}_\sigma$ for all grid-block centers and grid-block corners σ from the primary grid. This operator is obviously continuous and we have:

$$\|v\|_{1,h} \leq \sqrt{2}\, \|\tilde{v}\|_{2,h} \qquad \forall\, \tilde{v} \in \mathbf{E}(\mathcal{M})$$

Indeed, for all $\tilde{v} \in \mathbf{E}(\mathcal{M})$, we have:

$$[\tilde{v}_{i,j} - \tilde{v}_{i+1,j}]^2 \leq 2 \left\{ [\tilde{v}_{i,j} - \tilde{v}_{i+\frac{1}{2},j}]^2 + [\tilde{v}_{i+\frac{1}{2},j} - \tilde{v}_{i+1,j}]^2 \right\}$$

$$[\tilde{v}_{i,j} - \tilde{v}_{i,j+1}]^2 \leq 2 \left\{ [\tilde{v}_{i,j} - \tilde{v}_{i,j+\frac{1}{2}}]^2 + [\tilde{v}_{i,j+\frac{1}{2}} - \tilde{v}_{i,j+1}]^2 \right\}$$

$$\left[\tilde{v}_{i+\frac{1}{2},j+\frac{1}{2}} - \tilde{v}_{i-\frac{1}{2},j+\frac{1}{2}}\right]^2 \leq 2 \left\{ [\tilde{v}_{i+\frac{1}{2},j+\frac{1}{2}} - \tilde{v}_{i,j+\frac{1}{2}}]^2 + [\tilde{v}_{i,j+\frac{1}{2}} - \tilde{v}_{i-\frac{1}{2},j+\frac{1}{2}}]^2 \right\}$$

etc ... $\diamondsuit$

PROPOSITION 5.9 (Stability result)
The weak-star approximate solution $\tilde{u}_h$ of the diffusion problem (1.1)-(1.2) satisfies the following inequality:

$$\|\tilde{u}_h\|_{2,h} \leq C\, \|f\|_{L^2(\Omega)}\,. \quad \diamondsuit$$

Proof. The basic idea is that the edge mid-point pressures should not be considered as simple auxiliary unknowns. Therefore one should deal with a new discrete problem made up of three groups of equations: (i) the first group involves the flux balance equations in the primary grid-blocks ($K_{i,j}$) (see subsection 2.1 for the way of deriving those equations), (ii) the second group corresponds to the flux balance equations in the dual grid-blocks ($K_{i+\frac{1}{2},j+\frac{1}{2}}$), (iii) the third group concerns the flux continuity equations across the primary grid-block interfaces. Recall that:

$$U_{cc} = \{u_{i,j}\}_{1\leq i,j\leq N} \qquad \text{and} \qquad U_{vc} = \left\{u_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\leq i,j\leq N-1} \tag{5.9}$$

are respectively the sub-vectors of grid-block center pressures and grid-block corner pressures (with respect to the primary grid). On the other hand, we denote U_{ep} the sub-vector whose components are the edge mid-point pressures i.e. $\left\{u_{i+\frac{1}{2},j}\right\}_{1\leq i\leq N-1,\, 1\leq j\leq N}$ and $\left\{u_{i,j+\frac{1}{2}}\right\}_{1\leq i\leq N,\, 1\leq j\leq N-1}$.

The matrix form of the new discrete problem (which of course includes edge mid-point pressures) writes as follows:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} U_{cc} \\ U_{vc} \\ U_{ep} \end{bmatrix} = \begin{bmatrix} F_{cc} \\ F_{vc} \\ 0 \end{bmatrix} \tag{5.10}$$

where

$$F_{cc} = \left\{ \int_{K_{i,j}} f dx \right\}_{1 \le i,j \le N} \qquad \text{and} \qquad F_{vc} = \left\{ \int_{K_{i+\frac{1}{2},\ j+\frac{1}{2}}} f dx \right\}_{1 \le i,j \le N} \tag{5.11}$$

Note that A_{11}, A_{22} and A_{33} are square matrices whose sizes are respectively equal to the number of components of U_{cc}, U_{vc} and U_{ep}.

We deduce from (5.10) that

$$[U_{cc} \quad U_{vc} \quad U_{ep}] \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} U_{cc} \\ U_{vc} \\ U_{ep} \end{bmatrix} = [U_{cc} \quad U_{vc} \quad U_{ep}] \begin{bmatrix} F_{cc} \\ F_{vc} \\ 0 \end{bmatrix} \tag{5.12}$$

Let us set

$$LHS = [U_{cc} \quad U_{vc} \quad U_{ep}] \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} U_{cc} \\ U_{vc} \\ U_{ep} \end{bmatrix}$$

Following the ideas developed in section 2 and using the flux continuity across dual grid-block interfaces leads to

$$\gamma \, \|\tilde{u}_h\|_{2,h}^2 \le LHS \tag{5.13}$$

where γ is a strictly positive constant depending exclusively on the lithology structure of the porous medium Ω.

Remarking that

$$[U_{cc} \quad U_{vc} \quad U_{ep}] \begin{bmatrix} F_{cc} \\ F_{vc} \\ 0 \end{bmatrix} = [U_{cc} \quad U_{vc}] \begin{bmatrix} F_{cc} \\ F_{vc} \end{bmatrix}$$

we easily see that Proposition 5.9 follows from the proof of the stability of the weak approximate solution (see the proof of Proposition 4.4) and Remark 5.8. ∎

REMARK 5.10 It follows from the preceding proof (see 5.13) that the matrix

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{5.14}$$

associated with the new discrete problem is positive definite. ◊

We deal now with the error estimate for the weak-star approximate solution in the following discrete energy norm $\|\cdot\|_{2,h}$. For this purpose, we define the sub-vectors $\tilde{\varepsilon}_{h,cc}$, $\tilde{\varepsilon}_{h,vc}$ and $\tilde{\varepsilon}_{h,ep}$ as follows:

$$(\tilde{\varepsilon}_{h,cc})_{i,j} = \varphi_{i,j} - u_{i,j} \qquad (\tilde{\varepsilon}_{h,vc})_{i+\frac{1}{2},j+\frac{1}{2}} = \varphi_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i+\frac{1}{2},j+\frac{1}{2}} \tag{5.15}$$

$$(\tilde{\varepsilon}_{h,ep})_{i+\frac{1}{2},j} = \varphi_{i+\frac{1}{2},j} - u_{i+\frac{1}{2},j}, \qquad (\tilde{\varepsilon}_{h,ep})_{i,j+\frac{1}{2}} = \varphi_{i,j+\frac{1}{2}} - u_{i,j+\frac{1}{2}} \tag{5.16}$$

Using the same technique as the one developed in subsection 4.3 (and conserving all the previous notations), it is easy to check that the sub-vectors $\tilde{\varepsilon}_{h,cc}$, $\tilde{\varepsilon}_{h,vc}$ and $\tilde{\varepsilon}_{h,ep}$ satisfy a discrete system of the form (5.10). More precisely, we have:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon}_{h,cc} \\ \tilde{\varepsilon}_{h,vc} \\ \tilde{\varepsilon}_{h,ep} \end{bmatrix} = \begin{bmatrix} R_{h,cc} \\ R_{h,vc} \\ R_{h,ep} \end{bmatrix} \tag{5.17}$$

where we have set:

$$(R_{h,cc})_{i,j} = \sum_{e \in E_{i,j}} hR^e_{i,j} \qquad \forall\, 1 \le i,\ j \le N \tag{5.18}$$

$$(R_{h,vc})_{i+\frac{1}{2},j+\frac{1}{2}} = \sum_{e \in E_{i+\frac{1}{2},j+\frac{1}{2}}} hR^e_{i+\frac{1}{2},j+\frac{1}{2}} \quad \forall\, 1 \le i,\ j \le N-1 \tag{5.19}$$

$$(R_{h,ep})_{i+\frac{1}{2},j} = hR^e_{i,j} + hR^e_{i+1,j} \qquad \forall\, 1 \le i \le N-1 \qquad \forall\, 1 \le j \le N \tag{5.20}$$

$$(R_{h,ep})_{i,j+\frac{1}{2}} = hR^e_{i,j} + hR^e_{i,j+1} \qquad \forall\, 1 \le j \le N-1 \qquad \forall\, 1 \le i \le N \tag{5.21}$$

Recall that, if the exact solution φ is in C^2 over the closure of primary grid-blocks we have (see relations (4.15)):

$$\left| R^e_{i,j} \right| \le Ch \qquad \text{and} \qquad \left| R^e_{i+\frac{1}{2},j+\frac{1}{2}} \right| \le C\, h$$

We deduce from (5.17) that:

$$[\tilde{\varepsilon}_{h,cc}\, \tilde{\varepsilon}_{h,vc}\, \tilde{\varepsilon}_{h,ep}] \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon}_{h,cc} \\ \tilde{\varepsilon}_{h,vc} \\ \tilde{\varepsilon}_{h,ep} \end{bmatrix} = [\tilde{\varepsilon}_{h,cc}\, \tilde{\varepsilon}_{h,vc}\, \tilde{\varepsilon}_{h,ep}] \begin{bmatrix} R_{h,cc} \\ R_{h,vc} \\ R_{h,ep} \end{bmatrix} \tag{5.22}$$

Due to the positiveness definite of the matrix in the left hand side of (5.22), (see the preceding remark), we deduce that

$$\gamma \, \|\tilde{\varepsilon}_h\|^2_{2,h} \le [\tilde{\varepsilon}_{h,cc} \;\; \tilde{\varepsilon}_{h,vc} \;\; \tilde{\varepsilon}_{h,ep}] \begin{bmatrix} R_{h,cc} \\ R_{h,vc} \\ R_{h,ep} \end{bmatrix} \tag{5.23}$$

after having set

$$\tilde{\varepsilon}_h = [\tilde{\varepsilon}_{h,\,cc} \;\; \tilde{\varepsilon}_{h,\,vc} \;\; \tilde{\varepsilon}_{h,\,ep}]^t$$

Remarking that all the components from the sub-vectors $\tilde{\varepsilon}_{h,vc}$ and $\tilde{\varepsilon}_{h,ep}$ associated with boundary nodes are equal to zero, $\tilde{\varepsilon}_h$ may be seen as a function of $E(\mathcal{M})$. Following the ideas developed in subsection 4.3 and using Remark 5.8 leads straightly to

$$\|\tilde{\varepsilon}_h\|_{2,h} \le \tilde{\Lambda}\, h \tag{5.24}$$

Let us investigate the $L^\infty(\Omega)$ estimate of the function $\tilde{\varepsilon}_h$. For the sake of simplicity of notations, we denote $\tilde{\varepsilon}_n$ the value of $\tilde{\varepsilon}_h$ at the node $n \in \mathcal{N}$, with naturally $\tilde{\varepsilon}_n = 0$ if n is a boundary node. We introduce Δ_n the horizontal line (one may choose the vertical line) including the node n. Since $\tilde{\varepsilon}_n = 0$ for boundary nodes, it is clear that

$$\begin{aligned} |\tilde{\varepsilon}_n|^2 &\leq \left\{ \sum_{p\,,\,q\,\in\,\Delta_n,\ d(p,q)=\frac{h}{2}} |\tilde{\varepsilon}_p - \tilde{\varepsilon}_q| \right\}^2 = \\ &= \left\{ \sum_{p\,,\,q\,\in\,\Delta_n,\ d(p,q)=\frac{h}{2}} \sqrt{\frac{h}{2}} \frac{|\tilde{\varepsilon}_p - \tilde{\varepsilon}_q|}{\sqrt{\frac{h}{2}}} \right\}^2 \qquad \forall\, n \in \mathcal{N} \end{aligned}$$

Applying Cauchy-Schwarz inequality leads to:

$$\begin{aligned} |\tilde{\varepsilon}_n|^2 &\leq \frac{2}{h} \left\{ \sum_{p\,,\,q\,\in\,\Delta_n,\ d(p,q)=\frac{h}{2}} |\tilde{\varepsilon}_p - \tilde{\varepsilon}_q|^2 \right\} \qquad \forall\, n \in \mathcal{N} \\ &\leq \frac{2}{h} \left\{ \sum_{p\,,\,q\,\in\mathcal{N},\ d(p,q)=\frac{h}{2}} |\tilde{\varepsilon}_p - \tilde{\varepsilon}_q|^2 \right\} = \frac{2}{h}\, \|\tilde{\varepsilon}_h\|_{2,h}^2 \end{aligned} \tag{5.25}$$

Let us summarize the previous developments in the following formulation.

Theorem 5.11 (Error estimates in following norms: $\mathbf{L}^\infty(\Omega)$ and $\|.\|_{2,h}$)
Assume that the diffusion tensor D governing the diffusion problem (1.1)-(1.2) is a full matrix which is symmetric and positive definite, with piecewise constant coefficients. Assume also that the unique variational solution φ of (1.1)-(1.2) satisfies the condition $\varphi_{|\overline{K}} \in C^2\left(\overline{K}\right)$, for any grid-block K of the primary grid. Let us consider the space $\mathbf{E}(\mathcal{M})$ made up of functions v defined almost everywhere in Ω such that v is constant in each square element from the mesh $\mathcal{M}$ and zero elsewhere. Let us recall that $\tilde{u}_h = \tilde{\Pi} U^h_{\mathcal{S}} = \tilde{\Pi} U^h_{\mathcal{P}}$, and set $\tilde{\varphi}_h = \tilde{\Pi}\varphi$, where $\tilde{\Pi}$ is an operator introduced in sub-section 5.2.

Then, the function $\tilde{\varepsilon}_h = \tilde{\varphi}_h - \tilde{u}_h$ satisfies the following inequalities:

$$\begin{aligned} (i) \qquad & \|\tilde{\varepsilon}_h\|_{2,h} \leq C\,h \\ (ii) \qquad & \|\tilde{\varepsilon}_h\|_{L^\infty(\Omega)} \leq \sqrt{2}\,C\,h^{\frac{1}{2}} \end{aligned}$$

where C is a strictly positive real number depending exclusively on φ, Ω and γ. $\Diamond$

Corollary 5.12 (Error estimate in $L^2(\Omega) - norm$)
ε_h satisfies the following inequality

$$\|\tilde{\varepsilon}_h\|_{L^2(\Omega)} \leq C\,h\cdot$$

where C is a strictly positive constant independent of h. $\Diamond$

Proof. It is essentially based upon the first estimate given in the previous theorem and the following obvious result (discrete version of Poincaré inequality):

Lemma 5.13 (discrete version of Poincaré inequality)
There exists a strictly positive number $\tilde{P}$, without dependence on h, such that

$$\|\tilde{v}\|_{L^2(\Omega)} \leq \tilde{P}\ \|\tilde{v}\|_{2,h} \qquad \forall\ \tilde{v} \in \mathbf{E}(\mathcal{M}). \quad \Diamond$$

Here ends the proof of the corollary. ■

We have gathered within Theorem 5.11 and Corollary 5.24 all the ingredients for formulating error estimates related to the piecewise bilinear and the piecewise biquadratic approximate solutions of the diffusion problem (1.1)-(1.2). To prove these error estimates, the same technique as the one developed in Sub-section 5.1 may be used. Note that as in Sub-section 5.1, the following well-known relations play a key role when carrying out diverse proofs.

$$0 \leq \Phi_n(x) \leq 1 \qquad \text{and} \qquad \sum_{n\in N} \Phi_n(x) = 1 \qquad \forall\ x \in \overline{\Omega}$$

where Φ_n is the generic name of classical basis functions from either $\mathcal{Q}_1$ or $\mathcal{Q}_2$ finite element theory.

In this sub-section, we are concerned only with the anisotropic heterogeneous case, i.e. D is supposed to be a piecewise constant permeability tensor. Recall that the discontinuities of D are located over grid-block interfaces of the primary mesh.

Proposition 5.14 Under the same assumptions as those of Theorem 5.11, the exact solution φ and either the piecewise bilinear approximate solution $U^h_{\mathcal{S}}$ or the piecewise biquadratic approximate solution $U^h_{\mathcal{P}}$ satisfy the following estimates:

$$\| \wedge \varphi - U^h_S \|_{0,(\Omega)} \leq C\, h \tag{5.26}$$

$$\| \wedge \varphi - U^h_S \|_{L^\infty(\Omega)} \leq C\, h^{\frac{1}{2}} \tag{5.27}$$

$$\| \wedge \varphi - U^h_P \|_{0,(\Omega)} \leq C\, h \tag{5.28}$$

$$\| \wedge \varphi - U^h_P \|_{L^\infty(\Omega)} \leq C\, h^{\frac{1}{2}}. \quad \Diamond \tag{5.29}$$

It follows from Lemma 5.1, Lemma 5.2 and the previous proposition that:

Proposition 5.15 Under the same assumptions as those of Theorem 5.11, the exact solution φ and either the piecewise bilinear approximate solution $U^h_{\mathcal{S}}$ or the piecewise biquadratic approximate solution $U^h_{\mathcal{P}}$ satisfy the following estimates:

$$\left\| \varphi - U^h_{\mathcal{S}} \right\|_{0,\Omega} \leq C\, h \tag{5.30}$$

$$\left\| \varphi - U^h_{\mathcal{P}} \right\|_{0,\Omega} \leq C\, h. \quad \Diamond \tag{5.31}$$

Concluding remarks: Error estimates for the piecewise bilinear and the piecewise biquadratic approximate solutions have not been analyzed in the case of anisotropic homogeneous porous media. The reason is that the technique of computation of fluxes over grid-block boundaries (from the primary and the secondary grids) leads to an expression which does not involve edge mid-point pressures (see relations (2.6)-(2.7) in Remark 2.1). However one can apply the mid-point rule for computing the edge mid-point pressures, and this leads to the following error expressions (see Remark 2.1):

$$\varphi_{i,j+\frac{1}{2}} - u_{i,j+\frac{1}{2}} = O(h^{\frac{3}{2}})$$

and

$$\varphi_{i+\frac{1}{2},j} - u_{i+\frac{1}{2},j} = O(h^{\frac{3}{2}})$$

Therefore,

$$(\Lambda\varphi - U^h_{\mathcal{S}})(x) = \sum_{n\,\in\,\mathcal{N}} \tilde{\varepsilon}_n \Phi_n(x)$$

and

$$(\Lambda\varphi - U^h_{\mathcal{P}})(x) = \sum_{n\,\in\,\mathcal{N}} \tilde{\varepsilon}_n \Phi_n(x)$$

where $\tilde{\varepsilon}_n = \varphi_n - u_n$ is the error at the node n and obeys to $\tilde{\varepsilon}_n = O(h^{\frac{3}{2}})$ (accounting with Remark 4.9), and where $\mathcal{N}$ is the set of all the nodes i.e. the set of grid-block centers, grid-block corners and edge mid-points (with respect to the primary grid). Since

$$\sum_{n\,\in\,\mathcal{N}} |\Phi_n(x)| = \sum_{n\,\in\,\mathcal{N}} \Phi_n(x) = 1 \qquad\qquad \forall\ x \in \overline{\Omega}$$

we deduce that

$$\| \wedge \varphi - U^h_S \|_{L^\infty(\Omega)} \le C\, h^{\frac{3}{2}} \tag{5.32}$$

and

$$\| \wedge \varphi - U^h_P \|_{L^\infty(\Omega)} \le C\, h^{\frac{3}{2}}. \tag{5.33}$$

It follows from Lemma 5.1, Lemma 5.2 and the inequalities (5.32)-(5.33) that (Ω is supposed bounded, so $L^\infty(\Omega)$ is continuously imbedded in $L^2(\Omega)$):

$$\left\| \varphi - U^h_{\mathcal{S}} \right\|_{0,\Omega} \le C\, h^{\frac{3}{2}} \tag{5.34}$$

$$\left\| \varphi - U^h_{\mathcal{P}} \right\|_{0,\Omega} \le C\, h^{\frac{3}{2}}. \tag{5.35}$$

These last estimates ((5.34) and (5.35)) are probably suboptimal and could be improved. Indeed an error estimate of higher order has been obtained (see Proposition 5.4) with the piecewise linear approximate solution which involves less degrees of freedom than the piecewise bilinear and the piecewise biquadratic approximate solutions. Recall that in this work C denotes miscellaneous constants without dependence on h. $\Diamond$

6 Conclusions an perspectives

We have presented in this work the formulation of an MPFA finite volume scheme for flow problems in anisotropic heterogeneous porous media. For carrying out the analysis of the theoretical properties of different classes of continuous approximate solutions (i.e. linear, bilinear and biquadratic approximate solutions) connected to our MPFA formulation and introduced in Section 3, the following ingredients were needed: (i) Introduction of notions of weak and weak-star approximate solutions, (ii) Introduction of two discrete energy norms, namely $\|\cdot\|_{1,h}$ and $\|\cdot\|_{2,h}$. Note that the well posedness of the discrete problem was not an obvious issue and was solved affirmatively. The stability and the convergence of those weak and weak-star approximate solutions in $L^2(\Omega) - norm$ and $L^\infty(\Omega) - norm$ have been derived from estimates in discrete energy norms. Thanks to some classical results from Lagrange interpolation theory, and error estimates for weak and weak-star approximate solutions, we have derived error estimates for different classes of continuous approximate solutions in $L^2(\Omega) - norm$ and $L^\infty(\Omega) - norm$.

This MPFA method has been implemented successfully on distorted meshes for solving some Hydrodynamics problems from Benchmark challenges, namely Andra Couplex 1 model found in the literature, and the Benchmark session proposed in the frame of FVCA5 (Fifth Edition of the International Conference on Finite Volumes for Complex Applications). This is the reason why its [illegible] [illegible] is involved in our research program.

7 Bibliography

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