RATIONAL APPROXIMATION IN THE SENSE OF KATO FOR TRANSPORT SEMIGROUPS*

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Abstract. In this paper we mix the rational approximation procedure, which is a time approximation with approximation in the sense of Kato, which is a space approximation for linear transport equation. In 1970, H. J. Hejtmanek [9] gave such a procedure for approximation of the linear transport equation and he proved the convergence only for explicit Euler scheme. We extend this procedure to explicit and implicit Euler, Crank-Nicolson and Predictor-Corrector schemes which have the rate 1,2 and 3 in the sense of rational approximation. Finally, we construct the numerical illustration for justifying the above rate of convergence.

Key words. Acceptable rational function. Euler's explicit and implicit algorithms. Crank-Nicolson scheme. Predictor-Corrector algorithm. Rate of convergence. Free, absorption, production transport semigroups

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1. Introduction . Let X,Y be real or complex Banach spaces. Let $\|.\|$ denote the norm in X. $\mathcal{L}(X,Y)$ is the space of all bounded linear operators from X to Y, $\mathcal{L}(X) := \mathcal{L}(X,X)$. Let A be a closed densely defined linear operator in a Banach space X which generates a strongly continuous semigroup e^{tA}. By a rational approximation we mean the existence of a rational function $R(z), z \in \mathbb{C}$ such that $[R(\frac{t}{n}A)]^n$ tends in some sense to e^{tA}. It is clair that any rational function cannot have a such property, so we define

Definition 1.1. *A rational complex function R, is **acceptable**, if*

(i) $|R(z)| \leq 1$, *for all* $Re(z) \leq 0$;
(ii) $R(\mathrm{i}x) \neq 0$ *for all* $x \in \mathbb{R}$;
(iii) *There exists a real constant $p \geq 1$ such that $R(z) = e^z + O(|z|^{p+1})$ as $|z| \to 0$.*

In this definition the condition (iii) implies that $R(0) = R'(0) = 1$ and p is called the convergence rate of this approximation. If we want to emphasis on the rate of convergence we say that R, is ***p-acceptable*** (see [6]).

Concerning the approximation in time (semi-discrete approximation), there is wealth of literature concerning the convergence and stability of the rational approximations of an abstract Cauchy problem (see [1, 2, 3, 7, 10, 12, 13, 14, 15, 17]) In [10], Hersh and Kato have shown that if R is p-acceptable, then for any $f \in D(A^{p+2})$,

$$\lim_{n\to\infty} \|R(\frac{t}{n}A)^n f - e^{tA}f\| = 0 \tag{1.1}$$

and the rate of convergence is $O((1/n)^p)$.

In [2] the assertion (1.1) is improved by Brenner and Thomée in the following manner:

Theorem 1.2. *Let R be a p-acceptable rational function, then for any $f \in D(A^{p+1})$,*

$$\|R(\frac{t}{n}A)^n f - e^{tA}f\| = O((1/n)^{p+1}). \tag{1.2}$$

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In other words, the rate of convergence is p.

In [7] and [13] an important improvement concerning the rate of convergence is given in the case where A is the generator of an analytic strongly continuous semigroup, by proving that in this case one has (1.1) for any $f \in X$. In [14] and more recently in [15] and [17], the same problem is investigated, when A is the generator of an analytic semigroup and the time step size is not uniform. In [8] this problem is generalized in the case where the time step size is not uniform and A generates a C_0-semigroup which is not analytic and it is proved that for any $\alpha > 1/2$ and for every $s \in (0, \alpha - 1/2)$, there exists some constant C_* depending on α and s such that

$$||(R(\frac{t}{n}A)^n - e^{tA})(1 - A)^{-\alpha}|| \leq C_*(t+1)^{\frac{3}{2}}(\frac{t}{n})^{\beta}, \tag{1.3}$$

where $\beta = \frac{ps}{p+s+1}$.

In the next section we will give some different expressions of the rational approximation function in an abstract setting and we define few well known algorithms such as Euler explicit and implicit methods which have the same rate of convergence $p = 1$, Crank-Nicolson method with $p = 2$ and predictor corrector with $p = 3$. The corresponding rate of convergence of these methods in time follows from Theorem 1.2.

Concerning the approximation in space, when A is the generator of a (C_0) semigroup, we define the convergence in the sense of Kato (see [11]).

Definition 1.3. *We say that a sequence of Banach spaces* $\{(X_n, \|.\|_n) \ : \ n = 1, 2, \cdots\}$ *converges to a Banach space* $(X, \|.\|)$ *in the sense of Kato and we write*

$$X_n \xrightarrow{K} X$$

if for any n *there is a linear operator* $P_n \in \mathcal{L}(X, X_n)$ *(called an approximating operator) satisfying the following two conditions:*

(K1) $\lim_{n\to\infty} \|P_n f\|_n = \|f\|$ *for any* $f \in X$*;*

(K2) *for any* $f_n \in X_n$*, there exists* $f^{(n)} \in X$ *such that* $f_n = P_n f^{(n)}$ *with* $\|f^{(n)}\| \leq C\|f_n\|_n$ *(*C *is independent of* n*).*

Let $X_n \xrightarrow{K} X$*,* $B_n \in \mathcal{L}(X_n)$ *and* $B \in \mathcal{L}(X)$*. We say that* B_n *converges to* B *in the sense of Kato and we write* $B_n \xrightarrow{K} B$ *if*

$$\lim_{n\to\infty} \|B_n P_n f - P_n B f\|_n = 0 \tag{1.4}$$

for any $f \in X$.

In the above context T. Ushijima [16] recovered the Lax equivalence Theorem. Another investigation in this direction is accomplished in [5] by the two first authors of this paper in which it is constructed an approximation family for the transport semigroup which converges in the sense of Kato to transport semigroup.

We assume that R is is a p-admissible rational function, with

$$R(z) := \frac{P(z)}{Q(z)} = \frac{\sum_{j=0}^{k} \alpha_j z^j}{\sum_{j=0}^{\ell} \beta_j z^j} \tag{1.5}$$

Definition 1.4. *Let* A *be the generator of a* (C_0) *semigroup* $U(t)$*. We say that* R*, is* p***-acceptable in the sense of Kato****, if for any* $n \in \mathbb{N}$ *there exists a finite* *sequence of operators* $A_n^{(j)},\ j=1,2,\cdots,m=(k(k+1)\ell(\ell+1))/2$*, such that each of them is a finite difference approximation of* A *and*

$$\|U_n(t)P_nf - P_nU(t)f\|_n = O((1/n)^{p+1}), \tag{1.6}$$

where

$$U_n(t) = \frac{\alpha_0 I + \sum_{j=1}^{k}\alpha_j \prod_{p=1}^{j} A_n^{(p)}}{\beta_0 I + \sum_{j=1}^{\ell}\beta_j \prod_{p=1}^{j} A_n^{(p+k)}}. \tag{1.7}$$

We think that is difficult to establishes the existence of the sequence $A_n^{(j)},\ j=1,2,\cdots,m$ in a systematic manner, but we do believe that is possible to construct this sequence in case by case.

For an illustration we look to the linear transport equation. We consider a matter of particles, constituted of neutrons, electrons, ions and photons. Each particle moves on a straight line with constant velocity until it collides with another particle of the supporting medium resulting in absorption, scattering or multiplication. The unknown of the transport equation is the particle density function $u(x,v,t)$. This is a function in the phase space $(x,v)\in\Omega\times V\subset\mathbb{R}^{2n}$ at the time $t\geq 0$, which belongs to its natural space $X=L^1(\Omega,V)$. Actually, $\int_{\Omega\times V}u(x,v,t)\,dx\,dv$ designates the total number of particles in the whole space $\Omega\times V$ at the time t. The general form of the transport problem is the following

$$(\mathbf{TP})\qquad \begin{cases} \frac{\partial u}{\partial t} = -\mathbf{v}\cdot\nabla u - \sigma(x,v)u + \int_V p(x,v',v)u(x,v',t)dv' & \text{in } \Omega\times V,\\ u(x,v,t)=0 \qquad \text{if } x\cdot v<0, \quad \text{for all } x\in\partial\Omega \\ u(x,v,0)=f(x,v)\in X, \end{cases}$$

In this equation which is known as linear Boltzmann equation the first term of the right hand side $-v\cdot\nabla u(x,v,t)$ illustrates the movement of the classical group of the particles in the absence of the absorption and production interactions. The second term in which σ is the rate of absorption, represents the lost of the particles caused by the diffusion or absorption at the point (x,v) in the phase space. Finally the integral of the last term represents the production of the particles at the point (x,v) in the phase space. The kernel $p(x,v',v)$ in this integral generates the transition of the states of particles at the position x and having the velocity v' to the particles at the same position having the velocity v. The velocity space V is in general a spherical shell in $\mathbb{R}^n$, namely

$$V=\{v\in\mathbb{R}^n : 0\leq v_{\min}\leq |v|\leq v_{\max}\leq +\infty\}.$$

In this article, we study the particular feature of the transport equation in which we replace Ω with $(-a,a)$ and we take $V:=[-1,1]$. We assume that σ is a strictly positive continuous function with

$$0<s_m\leq\sigma(x)\leq s_M \text{ for almost any } x\in(-a,a) \tag{1.8}$$

and we replace the kernel $p(x,v,v')$ by $\frac{1}{2}p(x)$ which is a positive continuous function independent of (v,v'), such that

$$0<\sup_{x\in[-a,a]}p(x)=k_M. \tag{1.9}$$

With these assumptions the transport problem **(TP)** can be replaced by the following simplified problem

$$\begin{cases} \frac{\partial u}{\partial t} = -v\cdot\nabla u - \sigma(x)u + \frac{1}{2}\int_{-1}^{1} p(x)u(x,v,t)dv \quad \text{in } (-a,a)\times[-1,1]; \\ u(-a,v\geq 0,t)=0, \quad u(a,v\leq 0,t)=0 \quad \text{for all } t>0; \\ u(x,v,0)=f(\mathbf{x},v)\in L^1((-a,a)\times[-1,1]). \end{cases} \tag{1.10}$$

Remark 1.5.

(a) We denote the production term $Af = \frac{1}{2}\int_{-1}^{1} p(x)f(x,v)dv = p(x)Pf$, with

$$Pf = \frac{1}{2}\int_{-1}^{1} f(x,v)dv, \tag{1.11}$$

which is a rank one projection on $L^1((-a,a)\times[-1,1])$. This space being generating we get $\|P\|=1$, and $\|A\|=k_M$, since $\|A\|\leq k_M$ and for the constant function $p(x)=k_M$ we get the equality.

(b) It is well-known that the problem **(TP)** generates a (C_0) semigroup $U(t)$

For defining the approximating spaces X_n we proceed as in [5]. We divide the phase space $(-a,a)\times[-1,1]$ into a finite number of cells by chopping the x interval $(-a,a)$ into $2m_n$ equal parts and the v interval $[-1,1]$ into $2\mu_n$ equal parts; h_n and k_n are the lengths of these parts, that is,

$$h_n = \frac{a}{m_n}, \quad k_n = \frac{1}{\mu_n}.$$

Then each cell can be labeled by a pair of integers $(i,j)\in\mathcal{N}$, where

$$\mathcal{N} := \{(i,j) : i = -m_n,\dots,-1,0,1,\dots,m_n.\ j=-\mu_n,\dots,-1,0,1,\dots,\mu_n\}.$$

The number of the particles in cell $\gamma(i,j)=[ih_n,(i+1)h_n]\times[jk_n,(j+1)k_n]$ is written $\xi_{i,j}$.

We define the set of all real vectors $\xi_{i,j}$ as the Banach space X_n with the norm

$$\|\xi\|_n = \sum_{i,j}|\xi_{i,j}|, \quad \xi\in X_n\,.$$

In [5] for proving that the approximating space X_n converges in the sense of Kato to X, we have proved the following Lemma.

Lemma 1.6. *(See [5]) For $P_nf=\{\xi_{i,j} : (i,j)\in\mathcal{N}\}$ where*

$$\xi_{i,j} = \int_{ih_n}^{(i+1)h_n}\int_{jk_n}^{(j+1)k_n} f(x,v)\,dx\,dv,$$

we have

(i) $\|P_nf\|_n = \|f\|$, *for any* $0\leq f\in X$;

(ii) $\|P_n\|_{\mathcal{L}(X,X_n)} = 1$;

(iii) $\lim_{n\to\infty}\|P_nf\|_n = \|f\|$, *for any* $f\in X$.

The three last sections are concerned with different cases of transport equation. In the first one (section 3) we consider the collision free transport equation when the absorption rate σ and production p rate of transport problem **(TP)** are zero. We show that the approximating problem converges in the sense of Kato and by choosing an appropriate approximating operator for different schemes all the schemes (Explicit and implicit Euler, Crank-Nicolson and Predictor-Corrector) give a unique algorithm which is a discrete form of the exact solution. We have to point out that this is one of the rare partial differential equations such that by taking an adequate approximating operator for any scheme, one can retrieve a discrete version of the exact solution.

In the section 4 we take $\sigma \neq 0$ and $p \equiv 0$, the correspondent equation is called *tomography* or *absorbing transport* equation. Since here we cannot retrieve numerically the exact solution we prove that the rate of the explicit and implicit Euler, Crank-Nicolson and Predictor-Corrector schemes are respectively 1,2 and 3.

The section 5 is devoted to transport equation in his whole generality. In this case we cannot represent the explicit solution of the equation. So, we will use the Theorems 1.2 and 3.4 of [5] for proving the convergence of the approximate solution in the sense of Kato.

In the last section we construct the numerical illustration for justifying the above rate of convergence.

2. Finite-difference approximation in abstract setting. Let us consider the abstract Cauchy problem:

$$(\mathbf{CP}) \qquad \begin{cases} \frac{du}{dt} = Au & \text{for } t > 0, \\ u(0) = f \in X \end{cases}$$

in a Banach space X and assume that A is the generator of a bounded strongly continuous semigroup e^{tA} in X.

There are various methods for resolving this problem by time finite-difference approximation and the most well-known of them are

(a) ***Euler's implicit and explicit schemes:***

$$\frac{x_{n+1} - x_n}{\tau} = Ax_{n+1} \quad \text{and} \quad \frac{x_{n+1} - x_n}{\tau} = Ax_n,$$

which are equivalent to

$$x_{n+1} = (I - \tau A)^{-1} x_n \quad \text{and} \quad x_{n+1} = (I + \tau A)x_n.$$

Replacing τA by z the rational approximation function of Euler's implicit scheme becomes $R(z) = (1 - z)^{-1}$ and for explicit Euler's scheme $R(z) = 1 + z$.

(b) ***Crank-Nicolson scheme:***

The Crank-Nicolson scheme can be obtained by mixing the explicit and implicit Euler's schemes as follows. Take the $x_{n+1/2}$ the value of u at the point $t_{n+1/2}$ in the middle of $[t_n, t_{n+1}]$ such that

$$\frac{x_{n+1} - x_{n+1/2}}{\tau/2} = Ax_{n+1/2} \quad \text{and} \quad \frac{x_{n+1/2} - x_n}{\tau/2} = Ax_{n+1/2},$$

which gives

$$x_{n+1} = (I + (\frac{\tau}{2})A)(I - (\frac{\tau}{2})A)^{-1} x_n.$$

Here the rational approximation function will be $R(z) = (2 + z)(2 - z)^{-1}$.

(c) ***Predictor-Corrector scheme:***

Here we add the equation

$$\frac{x_{n+1}-x_n}{\tau} = A\left(\frac{x_{n+1}+x_n}{2}\right)$$

with a predicted equation

$$\frac{x_{n+1}-x_n}{\tau} = A(x_{n+1/2}),$$

where the predicted value of $x_{n+1/2}$ can be corrected by the equation

$$\frac{x_{n+1}-x_{n+1/2}}{\tau/2} = A(x_{n+1}).$$

This manipulation gives

$$x_{n+1} = x_n + \frac{\tau}{3}\left[Ax_n + A\left(2x_{n+1} - \frac{\tau}{2}Ax_{n+1}\right)\right]$$

and by separating x_{n+1} from x_n we get

$$x_{n+1} = (I + \frac{\tau}{3}A)(I - \frac{2\tau}{3}A + \frac{\tau^2}{6}A^2)^{-1}x_n.$$

The corresponding rational function would be $R(z) = (1+\frac{z}{3})(1-\frac{2z}{3}+\frac{z^2}{6})^{-1}$. We will see that the above representations of rational approximation functions of different schemes can be matched into the following definition for different values of the integer p.

Remark 2.1. For implicit Euler's method

$$R(z) = (1-z)^{-1} = 1 + z + O(z^2). \tag{2.1}$$

Since $|R(z)| = 1/[(1-Re(z))^2 + Im(z)^2]$ and $Re(z) \le 0$, we obtain the assertion (i) of Definition 1.1. For $z = \mathrm{i}x, R(z) = 1/(1-\mathrm{i}x) \ne 0$ and the assertion (ii) follows. Finally $\mathrm{e}^z = \sum_{k\ge 0} z^k/k!$ and (2.1) imply that

$$R(z) - \mathrm{e}^z = O(|z|^2), \tag{2.2}$$

the same estimation holds for Euler's explicit scheme and consequently we get the assertion (iii) of Definition 1.1, and by using Theorem 1.2 the rate of convergence is $p = 1$ for both implicit and explicit Euler methods.

Remark 2.2. For Crank-Nicolson method

$$R(z) = (2+z)(2-z)^{-1} = 1 + z + \frac{z^2}{2} + O(z^2). \tag{2.3}$$

Since for $a \le 0$, $\sqrt{(2-a)^2+b^2} \ge \sqrt{(2+a)^2+b^2}$, we obtain the assertion (i) of Definition 1.1. For $z = \mathrm{i}x, |R(z)| = |(2+\mathrm{i}x)/(2-\mathrm{i}x)| = 1 \ne 0$ and the assertion (ii) follows. Finally, $\mathrm{e}^z = \sum_{k\ge 0} z^k/k!$ and (2.3) imply that

$$R(z) - \mathrm{e}^z = O(|z|^3), \tag{2.4}$$

consequently we get the assertion (iii) of Definition 1.1 for $p = 2$, and by using Theorem 1.2 the rate of convergence is $p = 2$.

REMARK 2.3. For predictor-corrector scheme

$$R(z) = \frac{1 + \frac{z}{3}}{1 - \frac{2z}{3} + \frac{z^2}{6}} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + O(|z|^4). \tag{2.5}$$

We remark that for $z = \mathrm{i}x, 0 \neq |R(z)| \leq 1$, since $1+\frac{x^2}{9} \leq 1+\frac{x^2}{9}+\frac{x^4}{36}$. Furthermore the conformal transformation $z \mapsto \frac{2\mathrm{i}(z-1)}{|z-1|^2}+\mathrm{i}$, maps the left hand-side plane $[Rez \leq 0]$ into the unit disc $D(0,1)$, so according to maximum principle $|R(z)| \leq 1$ for all $Re(z) \leq 0$ and the assertions (i) and (ii) of Definition 1.1 follow. Finally, (2.5) implies that

$$R(z) - \mathrm{e}^z = O(|z|^4), \tag{2.6}$$

consequently we get the assertion (iii) of Definition 1.1 for $p = 3$, and by using Theorem 1.2 the rate of convergence is $p = 3$.

3. Approximation of collision-free transport equation. The first step in this model is when the particles move without obstacle, that is the medium is so rarefied such that there is no other particle can change the directions of each particle. In this case if at the time $t = 0$ and at the point x there are $f(x, v)$ particles with velocity v, then at the time t, these particles find themselves at the point $x - tv$. So that the solution of the collision-free transport problem

$$(\mathbf{CFTP}) \qquad \begin{cases} \frac{\partial u}{\partial t} = T_0 u := -v \cdot \nabla u \quad \text{in } \Omega \times V, \\ u(x, v, t) = 0 \qquad \text{if } x \cdot v < 0, \quad \text{for all } x \in \partial\Omega \\ u(x, v, 0) = f(x, v) \in X, \end{cases}$$

is given by the family of operators $\{U_0(t)\}_{t\in\mathbb{R}}$ defined by

$$u(x, v, t) = [U_0(t)f](x, v) := \begin{cases} f(x - tv, v), & \text{if } x - tv \in \Omega \\ 0 & \text{elsewhere} \end{cases} \tag{3.1}$$

which is called the collision free transport semigroup.

With this consideration the discrete version of the collision free semigroup defined in (3.1) will be

$$[U_{0,n}(k\tau_n)\xi]_{i,j} = \xi_{i-kj,j} \quad \text{if } (i, j) \in \mathcal{N} \text{ and } k = 1; \cdots, n. \tag{3.2}$$

In fact, given t, n and μ_n, we take $\tau_n = t/n$ and $m_n = [na(2\mu_n + 1) - t]/(2t)$ such that $\tau_n k_n/h_n = 1$, since $ih_n - \tau_n kjk_n = h_n(\imath - \frac{\tau_n k_n}{h_n} kj)$, so we get (3.2).

REMARK 3.1. *Here we adopt the convention that* $\xi_{i,j} = 0$, *whenever* $i < -(m_n+1)$ *or* $i > m_n$. *This takes care of the boundary condition that no particles enter* Ω *through* $\partial\Omega$.

THEOREM 3.2. *For* $U_0(t)f(x, v) = f(x - tv, v)$, *we have*

(a)

$$U_{0,n}(t)P_n f = P_n U_0(t)f. \tag{3.3}$$

(b) $\|P_n U_0(t)f\|_\infty = \sup\{|\xi_{i-nj,j}|$: *over all partitions* $\mathcal{N}\} \leq M$, *where the constant* M *is independent of* n.

From (3.3) *we get the convergence in the sense of Kato, with zero at the right hand side of* (1.6). *Proof.* The assertion **(a)** follows from

$$U_{0,n}(t)P_nf = U_{0,n}(t)\{\xi_{i,j}\} = \{\xi_{i-nj,j}\} = \left\{\int_{\gamma(i-nj,j)} f(x,v)dxdv\right\}$$
$$\left\{\int_{\gamma(i,j)} f(x-tv,v)dxdv\right\} = P_nU_0(t)f.$$

and the assertion **(b)** from

$$|\xi_{i-nj,j}| = \iint_{\gamma(i-nj,j)} f(x,v)dxdv \leq \|f\|.$$

□

By computation of this expression we follow the exact value of approximating solution. As we will see the collision-free transport equation is one of the seldom equations in which by the judicious choice of discretization operators, the final value of these methods coincide with the exact value of solution at the point of discretization.

For the sake of importance of this result, we will announce it as a Theorem

THEOREM 3.3. *Let us define for Euler's explicit, Euler's implicit, Crank-Nicolson and predictor-corrector schemes the following approximated semigroup:*

- $[U_{0,n}^{\texttt{Euler-exp}}(\tau_n)\xi]_{i,j} = [\xi + T_{0,n}^{\texttt{Euler-exp}}\xi]_{(i,j)}$, *where*

$$T_{0,n}^{\texttt{Euler-exp}} := -jk_n\tau_n\frac{\xi_{i,j}-\xi_{i-j,j}}{jh_n} = \tau_n\frac{k_n}{h_n}(\xi_{i-j,j}-\xi_{i,j}) \quad \textit{if } (i,j)\in\mathcal{N}, \tag{3.4}$$

- $[U_{0,n}^{\texttt{Euler-imp}}(\tau_n)\xi]_{i,j} = \left[(I - T_{0,n}^{\texttt{Euler-imp}})^{-1}\xi\right]_{i,j}$, *where*

$$T_{0,n}^{\texttt{Euler-imp}} := -jk_n\tau_n\frac{\xi_{i+j,j}-\xi_{i,j}}{jh_n} = (\xi_{i,j}-\xi_{i+j,j}) \quad \textit{if } (i,j)\in\mathcal{N}, \tag{3.5}$$

- $[U_{0,n}^{\texttt{Cr-Ni}}(\tau_n)\xi]_{i,j} = [(I+\frac{1}{2}T_{0,n}^{(1)})(I-\frac{1}{2}T_{0,n}^{(2)})^{-1}\xi]_{i,j}$, *where*

$$\begin{aligned}T_{0,n}^{(1)} :=& -jk_n\tau_n\frac{u(ih_n,jk_n) - u\left(\left(i-\frac{j}{2}\right)h_n,jk_n\right)}{jh_n/2}\\ &= 2\left(u\left((i-\frac{j}{2})h_n,jk_n\right) - u(ih_n,jk_n)\right).\end{aligned}$$

and

$$\begin{aligned}T_{0,n}^{(2)} :=& -jk_n\tau_n\frac{u\left(\left(i+\frac{j}{2}\right)h_n,jk_n\right) - u(ih_n,jk_n)}{jh_n/2}\\ &= 2\left(u(ih_n,jk_n) - u\left((i+\frac{j}{2})h_n,jk_n\right)\right)\end{aligned}$$

and finally

- $[U_{0,n}^{\texttt{pre-cor}}(\tau_n)\xi]_{i,j} = [(I+\frac{1}{3}\widetilde{T}_{0,n}^{(1)})(I-\frac{2}{3}\widetilde{T}_{0,n}^{(2)}+\frac{1}{6}\widetilde{T}_{0,n}^{(3)}\widetilde{T}_{0,n}^{(4)})^{-1}\xi]_{i,j}$, *where*

$$\widetilde{T}_{0,n}^{(1)} := 3\left[T_{0,n}^{\texttt{Euler-exp}}\xi\right]_{i,j} + \left[T_{0,n}^{\texttt{Euler-exp}}\xi\right]_{i-j,j} - 3\left[T_{0,n}^{\texttt{Euler-exp}}\xi\right]_{i-\frac{j}{2},j}$$
$$= \left[\xi_{i-2j,j} - 3\xi_{i-\frac{3j}{2},j} + 2\xi_{i-j,j} + 3\xi_{i-\frac{j}{2},j} - 3\xi_{i,j}\right],$$
$$\widetilde{T}_{0,n}^{(2)} = \widetilde{T}_{0,n}^{(3)} := \frac{1}{2}\left[T_{0,n}^{(1)}\xi + T_{0,n}^{(2)}\xi\right]_{i,j} = \left[\xi_{i-\frac{j}{2},j} - \xi_{i+\frac{j}{2},j}\right] \quad \textit{and} \quad \widetilde{T}_{0,n}^{(4)} := T_{0,n}^{(1)}.$$

Then we have

$$[U_{0,n}^{\texttt{Euler-exp}}(\tau_n)\xi]_{i,j} = [U_{0,n}^{\texttt{Euler-imp}}(\tau_n)\xi]_{i,j} = [U_{0,n}^{\texttt{Cr-Ni}}(\tau_n)\xi]_{i,j} = [U_{0,n}^{\texttt{pre-cor}}(\tau_n)\xi]_{i,j} = \xi_{i-j,j}.$$

Proof. For explicit Euler's scheme, by choosing m_n and μ_n such that $\tau_n\frac{k_n}{h_n} = 1$ for any n, a simple calculation gives the expression of (3.2) for $k = 1$.

For implicit Euler's scheme, since

$$\xi_{i,j} = [(I - T_{0,n}^{(1)})\eta]_{i,j} = \eta_{i,j} - \tau_n\frac{k_n}{h_n}(\eta_{i,j} - \eta_{i+j,j}) = \eta_{i+j,j}$$

and once more we obtain $\eta_{i,j} = \xi_{i-j,j}$, as we have expected.

For the Crank-Nicolson scheme, since

$$[U_{0,n}^{\texttt{Cr-Ni}}(\tau_n)\xi]_{i,j} = [(I+\frac{1}{2}T_{0,n}^{(1)})(I-\frac{1}{2}T_{0,n}^{(2)})^{-1}\xi]_{i,j}$$
$$= (I+\frac{1}{2}T_n^{(1)})\xi_{i-\frac{j}{2},j} = \xi_{i-j,j}.$$

Finally, for the predictor-corrector scheme we remark that

$$\left[(I-\frac{2}{3}\widetilde{T}_{0,n}^{(2)}+\frac{1}{6}\widetilde{T}_{0,n}^{(3)}\widetilde{T}_{0,n}^{(4)})\xi\right]_{i,j} = \xi_{i,j} - \frac{2}{3}(\xi_{i-\frac{j}{2},j} - \xi_{i+\frac{j}{2},j}) + \frac{1}{3}\widetilde{T}_{0,n}^{(3)}(\xi_{i-\frac{j}{2},j} - \xi_{i,j})$$
$$= \frac{2}{3}\xi_{i,j} + \frac{1}{3}\xi_{i-j,j} + \xi_{i+\frac{j}{2},j} - \xi_{i-\frac{j}{2},j}$$

and from other hand

$$\left[(I+\frac{1}{3}\widetilde{T}_{0,n}^{(1)})\xi\right]_{i,j} = \xi_{i,j} + \frac{1}{3}\Big[-3(\xi_{i,j}-\xi_{i-j,j}) - (\xi_{i-j,j}-\xi_{i-2j,j}) + 3(\xi_{i-\frac{j}{2},j} - \xi_{i-\frac{3j}{2},j})\Big]$$
$$= \frac{2}{3}\xi_{i-j,j} + \frac{1}{3}\xi_{i-2j,j} + \xi_{i-\frac{j}{2},j} - \xi_{i-\frac{3j}{2},j} = \left[(I-\frac{2}{3}\widetilde{T}_{0,n}^{(2)}+\frac{1}{6}\widetilde{T}_{0,n}^{(3)}\widetilde{T}_{0,n}^{(4)})\xi\right]_{i-j,j}.$$

This proves that we obtain once more (3.2) for $k = 1$. □

In the section 6 we illustrate numerically the evolution of the pure translation of an initial solution with non entrance boundary condition. Remark that all the different schemes end with an unique scheme (3.2).

4. One dimensional pure absorbing linear transport equation. In this section we will choose the same approximating space X_n of the section 3 with the same condition $\tau_n\frac{k_n}{h_n} = 1$ on the grid. The exact solution of the pure absorbing transport problem

$$(\mathbf{PATP}) \qquad \begin{cases} \frac{\partial u}{\partial t} = -v\cdot\nabla u - \sigma(x)u \quad \text{in } (-a,a)\times[-1,1]; \\ u(-a,v\geq 0,t) = 0 \text{ and } u(a,v\leq 0,t) = 0 \text{ for all } t>0; \\ u(x,v,0) = f(\mathbf{x},v) \in L^1((-a,a)\times[-1,1]). \end{cases}$$

is given by

$$u(x,v,t)=[U_1(t)f](x,v):=\begin{cases}\mathrm{e}^{-\int_0^t\sigma(x-sv)ds}f(x-tv,v), & \text{if} \quad |x-tv|<a\\ 0 & \text{elsewhere}\end{cases} \tag{4.1}$$

where $U_1(t)$ is a C_0-semigroup on X.

The one dimensional approximation of this solution would be

$$u(ih_n,jk_n,k\tau_n)=\exp\left(-\int_0^t\sigma(ih_n-sjk_n)ds\right)f(ih_n-jk\tau_nk_n,jk_n)$$

if $(i,j)\in\mathcal{N}$ and $k=1;\cdots,n$.

After replacing the integral $\int_0^t\sigma(ih_n-sjk_n)ds$ by $\boldsymbol{\sigma}_{i,j}^{(n)}$, where

$$\boldsymbol{\sigma}_{i,j}^{(l)}:=\tau_n\sum_{k=1}^{l}\sigma(ih_n-jk\tau_nk_n). \tag{4.2}$$

Then we get

$$U_{1,n}(t)=u(ih_n,jk_n,t)=\exp\left(-\boldsymbol{\sigma}_{i,j}^{(n)}\right)f(ih_n-nj\tau_nk_n,jk_n).$$

Replacing $f(ih_n-jn\tau_nk_n,jk_n)$ by $\xi_{i-nj,j}$ as before we get

$$[U_{1,n}(n\tau_n)\xi]_{i,j}=\exp\left(-\boldsymbol{\sigma}_{i,j}^{(n)}\right)\xi_{i-nj,j}. \tag{4.3}$$

Theorem 4.1. *We assume that σ is a strictly positive continuous function satisfying* (1.8) *and $U_1(t)$ defined in* (4.1)*, we have the convergence of $U_{1,n}(t)$ to $U_1(t)$ in the sense of Kato.* *Proof.* It is well known that if

$$s_n(\sigma)=\tau_n\sum_{k=1}^{n}m_k(\sigma),\quad S_n(\sigma)=\tau_n\sum_{k=1}^{n}M_k(\sigma),$$

where

$$m_k(\sigma)=\inf_{s\in[(k-1)\tau_n,k\tau_n]}\sigma(x-sv)\quad\text{and}\quad M_k(\sigma)=\sup_{s\in[(k-1)\tau_n,k\tau_n]}\sigma(x-sv),$$

were the upper and lower Darboux's sum of the function $[0,t]\ni s\mapsto\sigma(x-sv)\in\mathcal{C}([-a,a]\times[-1,1])$, then $s_n(\sigma)\le\boldsymbol{\sigma}_{i,j}^{(n)}\le S_n(\sigma)$ and $s_n(\sigma)$ and $S_n(\sigma)$ converge both to $\int_0^t\sigma(\mathbf{x}-sv)ds$ in $\mathcal{C}([-a,a]\times[-1,1])$. So

$$\|\exp\left(-\boldsymbol{\sigma}^{(n)}\right)-P_n\left(\mathrm{e}^{-\int_0^t\sigma(\mathbf{x}-sv)ds}\right)\|_n\to 0,$$

where $\boldsymbol{\sigma}^{(n)}\in X_n$ with $\left[\boldsymbol{\sigma}^{(n)}\right]_{i,j}=\sigma_{i,j}^{(n)}$. And, due to this property,

$$\begin{aligned}\|U_{1,n}(t)P_nf-P_nU_1(t)f\|_n&\le\|U_{1,n}(t)P_nf-\exp\left(-\boldsymbol{\sigma}^{(n)}\right)P_nU_0(t)f\|_n\\&\quad+\underbrace{\|\exp\left(-\boldsymbol{\sigma}^{(n)}\right)P_nU_0(t)f-P_nU_1(t)f\|_n}_{=0}\\&\le\|P_nU_0(t)f\|_\infty\|P_n\left(\mathrm{e}^{-\int_0^t\sigma(\mathbf{x}-sv)ds}\right)-\exp\left(-\tau_n\boldsymbol{\sigma}^{(n)}\right)\|_n,\end{aligned}$$

which goes to zero as $n\to\infty$. In fact, according to Theorem 3.2 **(b)** can be estimated independently of n. □

Now let us describe the different schemes for the problem **(PATP)**.

4.1. Euler's explicit and implicit scheme for pure absorbing linear transport equation. For explicit Euler's scheme we define the finite difference operator

$$T_{1,n}^{\texttt{Euler-exp}} := (\xi_{i-j,j} - \xi_{i,j}) - \tau_n \sigma_{i-j} \xi_{i-j,j} \quad \text{if } (i,j) \in \mathcal{N} \tag{4.4}$$

the approximated semigroup $U_{1,n}^{\texttt{Euler-exp}}(\tau_n)$ would be

$$[\eta]_{i,j} := [U_{1,n}^{\texttt{Euler-exp}}(\tau_n)\xi]_{i,j} = [\xi + T_{1,n}^{\texttt{Euler-exp}}\xi]_{i,j} = \xi_{i,j} + ((\xi_{i-j,j} - \xi_{i,j}) - \tau_n \sigma_{i-j}\xi_{i-j,j}) \,.$$

and so,

$$[\eta]_{i,j} = (1 - \tau_n \sigma_{i-j})\xi_{i-j,j}.$$

A comparison with respect to the pure absorption approximate group (4.3) leads the following estimation:

$$|[(U_{1,n}^{\texttt{Euler-exp}}(\tau_n) - U_{1,n}(\tau_n))\xi]_{i,j}| = |1 - \tau_n\sigma_{i-j} - \exp(-\tau_n\sigma_{i-j})\,||\xi_{i-j,j}| = O(\tau_n^2)|\xi_{i-j,j}|,$$

which implies that there exists a constant C depending only on σ, such that

$$\|(I + T_{1,n}^{\texttt{Euler-exp}} - U_{1,n}(\tau_n))\xi\|_n \leq C\tau_n^2 \|\xi\|_n.$$

which leads to the estimation (2.2) and consequently the order of the scheme would be $p = 1$.

For implicit Euler's scheme the finite difference operator would be

$$T_{1,n}^{\texttt{Euler-imp}} := (\xi_{i,j} - \xi_{i+j,j}) - \tau_n \sigma_i \xi_{i+j,j} \quad \text{if } (i,j) \in \mathcal{N} \tag{4.5}$$

and $U_{1,n}^{\texttt{Euler-imp}}(\tau_n)$ by

$$[\eta]_{i,j} = [U_{1,n}^{\texttt{Euler-imp}}(\tau_n)\xi]_{i,j} := [\left(I - T_{1,n}^{\texttt{Euler-imp}}\right)^{-1}\xi]_{i,j}.$$

So,

$$\xi_{i,j} = [\eta - T_{1,n}^{\texttt{Euler-imp}}\eta]_{i,j} = \eta_{i,j} - ((\eta_{i,j} - \eta_{i+j,j}) - \tau_n\sigma_i\eta_{i+j,j}) = \eta_{i+j,j} + \tau_n\sigma_i\eta_{i+j,j},$$

which gives,

$$[\eta]_{i,j} = (1 + \tau_n\sigma_{i-j})^{-1}\xi_{i-j,j}.$$

and consequently

$$|\left[\left(I - T_{1,n}^{\texttt{Euler-imp}}\right)^{-1}\xi - U_{1,n}(\tau_n)\xi\right]_{i,j}| = |(1 + \tau_n\sigma_{i-j})^{-1} - \exp(-\tau_n\sigma_{i-j})\,||\xi_{i-j,j}|$$

which implies once more

$$\|U_{1,n}^{\texttt{Euler-imp}}(\tau_n)\xi - U_{1,n}(\tau_n))\xi\|_n \leq C\tau_n^2\|\xi\|_n.$$

and consequently the order of the scheme would be $p = 1$.

4.2. Crank-Nicolson scheme for pure absorbing linear transport equation. For this scheme we define two finite difference operators $T_{1,n}^{(1)}$ and $T_{1,n}^{(2)} \in \mathcal{L}(X_n)$ as

$$[T_{1,n}^{(1)}\xi]_{i,j} := 2(\xi_{i-\frac{j}{2},j} - \xi_{i,j}) - \tau_n \sigma_{i-\frac{j}{2}} \xi_{i-\frac{j}{2},j}$$

and

$$[T_{1,n}^{(2)}\xi]_{i,j} := 2(\xi_{i,j} - \xi_{i+\frac{j}{2},j}) - \tau_n \sigma_{i+\frac{j}{2}} \xi_{i+\frac{j}{2},j}$$

We define also the following operator

$$[M\xi]_{i,j} = R(-\tau_n\sigma_i)\xi_{i,j} \quad \text{where} \quad R(z) = (2+z)(2-z)^{-1}$$

We remark that

$$\begin{aligned}[(1+\frac{1}{2}T_{1,n}^{(1)})\xi]_{i,j} &= \xi_{i,j} + (\xi_{i-\frac{j}{2},j} - \xi_{i,j}) - \frac{\tau_n}{2}\sigma_{i-\frac{j}{2}}\xi_{i-\frac{j}{2},j} \\ &= (1-\frac{\tau_n}{2}\sigma_{i-\frac{j}{2}})\xi_{i-\frac{j}{2},j}\end{aligned}$$

and

$$\begin{aligned}[(1-\frac{1}{2}T_{1,n}^{(2)})M\xi]_{i-j,j} &= [M\xi]_{i-j,j} - \Big([M\xi]_{i-j,j} - [M\xi]_{i-\frac{j}{2},j}\Big) + \frac{\tau_n}{2}\sigma_{i-\frac{j}{2}}[M\xi]_{i-\frac{j}{2},j} \\ &= (1+\frac{\tau_n}{2}\sigma_{i-\frac{j}{2}})[M\xi]_{i-\frac{j}{2},j} = (1-\frac{\tau_n}{2}\sigma_{i-\frac{j}{2}})\xi_{i-\frac{j}{2},j} = [(1+\frac{1}{2}T_{1,n}^{(1)})\xi]_{i,j}.\end{aligned}$$

This proves that

$$[\eta]_{i,j} = [(1+\frac{1}{2}T_{1,n}^{(1)})(1-\frac{1}{2}T_{1,n}^{(2)})^{-1}\xi]_{i,j} = [M\xi]_{i-j,j} = (2-\tau_n\sigma_{i-j})(2+\tau_n\sigma_{i-j})^{-1}\xi_{i-j,j}.$$

Now, if we define $U_{1,n}^{\mathtt{Cr-Ni}}(\tau_n)$ by

$$[U_{1,n}^{\mathtt{Cr-Ni}}(\tau_n)\xi]_{i,j} := [(1+\frac{1}{2}T_{1,n}^{(1)})(1-\frac{1}{2}T_{1,n}^{(2)})^{-1}\xi]_{i,j}$$

a comparison with the pure absorption approximate semigroup $U_{1,n}(t)$ leads the following estimation :

$$|[\eta - U_{1,n}(\tau_n)\xi]_{i,j}| = |(2-\tau_n\sigma_{i-j})(2+\tau_n\sigma_{i-j})^{-1} - \exp(-\tau_n\sigma_{i-j})||\xi_{i-j,j}|$$

which implies

$$\|U_{1,n}^{\mathtt{Cr-Ni}}(\tau_n)\xi - U_{1,n}(\tau_n)\xi\|_n \le C\tau_n^3||\xi||_n.$$

where the constant C depends only on σ, and independent of τ_n and we retrieve the estimation (2.4) and consequently the order of the scheme would be $p = 2$.

4.3. Predictor-corrector scheme for pure absorbing linear transport equation. Here we define four finite difference operators,

- $[\widetilde{T}_{1,n}^{(1)}\xi]_{i,j} := 3[T_{0,n}^{\mathtt{Euler-exp}}\xi]_{i,j} + [T_{0,n}^{\mathtt{Euler-exp}}\xi]_{i-j,j} - 3[T_{0,n}^{\mathtt{Euler-exp}}\xi]_{i-\frac{j}{2},j}$ $-\frac{2}{3}\tau_n\sigma_{i-j}\xi_{i-j,j} - \frac{1}{3}\tau_n\sigma_{i-2j}\xi_{i-2j,j} - \tau_n\sigma_{i-\frac{j}{2}}\xi_{i-\frac{j}{2},j} + \tau_n\sigma_{i-\frac{3}{2}j}\xi_{i-\frac{3}{2}j,j}$

- $[\widetilde{T}^{(2)}_{1,n}\xi]_{i,j} := (\xi_{i-\frac{j}{2},j} - \xi_{i+\frac{j}{2},j}) + \frac{1}{2}\tau_n\sigma_{i+\frac{j}{2}}\xi_{i+j,j} - \frac{3}{4}\tau_n\sigma_{i-j}\xi_{i-j,j} - \tau_n(\frac{3}{4}\sigma_i + \frac{1}{2}\sigma_{i-\frac{j}{2}})\xi_{i,j}$
$+ \tau_n(\frac{1}{3}\sigma_i - \frac{3}{2}\sigma_{i+\frac{j}{2}})\xi_{i+\frac{j}{2},j} - \tau_n(\frac{1}{6}\sigma_{i-j} + \frac{3}{2}\sigma_{i-\frac{j}{2}})\xi_{i-\frac{j}{2},j}$
- $[\widetilde{T}^{(3)}_{1,n}\xi]_{i,j} := (\xi_{i-\frac{j}{2},j} - \xi_{i+\frac{j}{2},j}) - \frac{2}{3}\tau_n\sigma_i\xi_{i+\frac{j}{2},j} - \frac{1}{3}\tau_n\sigma_{i-j}\xi_{i-\frac{j}{2},j} - \tau_n\sigma_{i+\frac{j}{2}}\xi_{i+j,j} + \tau_n\sigma_{i-\frac{j}{2}}\xi_{i,j}$
- $[\widetilde{T}^{(4)}_{1,n}\xi]_{i,j} = [T^{(1)}_{1,n}\xi]_{i,j}$.

We define also the following operator

$$[M_1\xi]_{i,j} = R_1(-\tau_n\sigma_i)\xi_{i,j} \quad \text{where} \quad R_1(z) = (1+\frac{z}{3})(1-\frac{2}{3}z+\frac{z^2}{6})^{-1}.$$

By a simple calculation, we obtain

$$\begin{aligned}
[(1+\frac{1}{3}\widetilde{T}^{(1)}_{1,n})\xi]_{i,j} &= \frac{2}{3}\xi_{i-j,j} + \frac{1}{3}\xi_{i-2j,j} + \xi_{i-\frac{j}{2},j} - \xi_{i-\frac{3}{2}j,j} - \frac{2}{9}\tau_n\sigma_{i-j}\xi_{i-j,j} \\
&\quad - \frac{\tau_n}{9}\sigma_{i-2j}\xi_{i-2j,j} - \frac{\tau_n}{3}\sigma_{i-\frac{j}{2}}\xi_{i-\frac{j}{2},j} + \frac{\tau_n}{3}\sigma_{i-\frac{3}{2}j}\xi_{i-\frac{3}{2}j,j} \\
&= \frac{2}{3}(1-\frac{\tau_n}{3}\sigma_{i-j})\ \xi_{i-j,j} + \frac{1}{3}(1-\frac{\tau_n}{3}\sigma_{i-2j})\xi_{i-2j,j} \\
&\quad + (1-\frac{\tau_n}{3}\sigma_{i-\frac{j}{2}})\xi_{i-\frac{j}{2},j} - (1-\frac{\tau_n}{3}\sigma_{i-\frac{3}{2}j})\xi_{i-\frac{3}{2}j,j}
\end{aligned}$$

and

$$[(1-\frac{2}{3}\widetilde{T}^{(2)}_{1,n} + \frac{1}{6}\widetilde{T}^{(3)}_{1,n}\widetilde{T}^{(4)}_{1,n})M_1\xi]_{i-j,j} = A_{i,j} + \frac{1}{6}B_{i,j}$$

where

$$\begin{aligned}
A_{i,j} &= [M_1\xi]_{i-j,j} - \frac{2}{3}([M_1\xi]_{i-\frac{3}{2}j,j} - [M_1\xi]_{i-\frac{j}{2},j}) - \frac{\tau_n}{3}\sigma_{i-\frac{j}{2}}[M_1\xi]_{i,j} \\
&\quad + \frac{\tau_n}{2}\sigma_{i-2j}[M_1\xi]_{i-2j,j} + \frac{\tau_n}{2}\sigma_{i-j}[M_1\xi]_{i-j,j} + \frac{\tau_n}{3}\sigma_{i-\frac{3}{2}j}[M_1\xi]_{i-j,j} \\
&\quad - \frac{2\tau_n}{9}\sigma_{i-j}[M_1\xi]_{i-\frac{j}{2},j} + \tau_n\sigma_{i-\frac{j}{2}}[M_1\xi]_{i-\frac{j}{2},j} - \frac{\tau_n}{9}\sigma_{i-2j}[M_1\xi]_{i-\frac{3}{2}j,j} \\
&\quad - \tau_n\sigma_{i-\frac{3}{2}j}[M_1\xi]_{i-\frac{3}{2}j,j}
\end{aligned}$$

and

$$\begin{aligned}
B_{i,j} &= \left([\widetilde{T}^{(4)}_{1,n}M_1\xi]_{i-\frac{3}{2}j,j} - [\widetilde{T}^{(4)}_{1,n}M_1\xi]_{i-\frac{j}{2},j}\right) - \frac{2}{3}\tau_n\sigma_{i-j}[\widetilde{T}^{(4)}_{1,n}M_1\xi]_{i-\frac{j}{2},j} \\
&\quad - \frac{1}{3}\tau_n\sigma_{i-2j}[\widetilde{T}^{(4)}_{1,n}M_1\xi]_{i-\frac{3}{2}j,j} - \tau_n\sigma_{i-\frac{j}{2}}[\widetilde{T}^{(4)}_{1,n}M_1\xi]_{i,j} + \tau_n\sigma_{i-\frac{3}{2}j}[\widetilde{T}^{(4)}_{1,n}M_1\xi]_{i-j,j} \\
&= \Big\{2\left([M_1\xi]_{i-2j,j} - [M_1\xi]_{i-\frac{3}{2}j,j} - [M_1\xi]_{i-j,j} + [M_1\xi]_{i-\frac{j}{2},j}\right) \\
&\quad - \tau_n\sigma_{i-2j}[M_1\xi]_{i-2j,j} + \tau_n\sigma_{i-j}[M_1\xi]_{i-j,j}\Big\} - \frac{2}{3}\tau_n\sigma_{i-j}\Big\{2\Big([M_1\xi]_{i-j,j} \\
&\quad - [M_1\xi]_{i-\frac{j}{2},j}\Big) - \tau_n\sigma_{i-j}[M_1\xi]_{i-j,j}\Big\} - \frac{1}{3}\tau_n\sigma_{i-2j}\Big\{2\left([M_1\xi]_{i-2j,j} - [M_1\xi]_{i-\frac{3}{2}j,j}\right) \\
&\quad - \tau_n\sigma_{i-2j}[M_1\xi]_{i-2j,j}\Big\} - \tau_n\sigma_{i-\frac{j}{2}}\left\{2\left([M_1\xi]_{i-\frac{j}{2},j} - [M_1\xi]_{i,j}\right) - \tau_n\sigma_{i-\frac{j}{2}}[M_1\xi]_{i-\frac{j}{2},j}\right\} \\
&\quad + \tau_n\sigma_{i-\frac{3}{2}j}\left\{2\left([M_1\xi]_{i-\frac{3}{2}j,j} - [M_1\xi]_{i-j,j}\right) - \tau_n\sigma_{i-\frac{3}{2}j}[M_1\xi]_{i-\frac{3}{2}j,j}\right\}
\end{aligned}$$

So we get

$$A_{i,j}+\frac{1}{6}B_{i,j}=\frac{2}{3}\left(1+\frac{2}{3}\tau_n\sigma_{i-j}+\frac{\tau_n^2}{6}\sigma_{i-j}^2\right)[M_1\xi]_{i-j,j}+\frac{1}{3}\Big(1+\frac{2}{3}\tau_n\sigma_{i-2j}+\frac{\tau_n^2}{6}\sigma_{i-2j}^2\Big)[M_1\xi]_{i-2j,j}+\left(1+\frac{2}{3}\tau_n\sigma_{i-\frac{j}{2}}+\frac{\tau_n^2}{6}\sigma_{i-\frac{j}{2}}^2\right)[M_1\xi]_{i-\frac{j}{2},j}-\left(1+\frac{2}{3}\tau_n\sigma_{i-\frac{3}{2}j}+\frac{\tau_n^2}{6}\sigma_{i-\frac{3}{2}j}^2\right)[M_1\xi]_{i-\frac{3}{2}j,j}$$

and finally,

$$[(1-\frac{2}{3}\widetilde{T}_{1,n}^{(2)}+\frac{1}{6}\widetilde{T}_{1,n}^{(3)}\widetilde{T}_{1,n}^{(4)})M_1\xi]_{i-j,j}=\frac{2}{3}(1-\frac{\tau_n}{3}\sigma_{i-j})\xi_{i-j,j}+\frac{1}{3}(1-\frac{\tau_n}{3}\sigma_{i-2j})\xi_{i-2j,j}+(1-\frac{\tau_n}{3}\sigma_{i-\frac{j}{2}})\xi_{i-\frac{j}{2},j}-(1-\frac{\tau_n}{3}\sigma_{i-\frac{3}{2}j})\xi_{i-\frac{3}{2}j,j}=[(1+\frac{1}{3}\widetilde{T}_{1,n}^{(1)})\xi]_{i,j}$$

which implies

$$[\eta]_{i,j}=[(1-\frac{2}{3}\widetilde{T}_{1,n}^{(2)}+\frac{1}{6}\widetilde{T}_{1,n}^{(3)}\widetilde{T}_{1,n}^{(4)}))^{-1}(1+\frac{1}{3}\widetilde{T}_{1,n}^{(1)})\xi]_{i,j}=[M_1\xi]_{i-j,j}=(1-\frac{\tau_n}{3}\sigma_{i-j})(1+\frac{2}{3}\tau_n\sigma_{i-j}+\frac{\tau_n^2}{6}\sigma_{i-j}^2)^{-1}\xi_{i-j,j}$$

and we get

$$|[\eta-U_{1,n}(\tau_n)\xi]_{i,j}|=|(1-\frac{\tau_n}{3}\sigma_{i-j})(1+\frac{2}{3}\tau_n\sigma_{i-j}+\frac{\tau_n^2}{6}\sigma_{i-j}^2)^{-1}-\exp(-\tau_n\sigma_{i-j})||\xi_{i-j,j}|.$$

So, if we define

$$U_{1,n}^{\texttt{pre-cor}}(\tau_n)=(1+\frac{1}{3}\widetilde{T}_{1,n}^{(1)})(1-\frac{2}{3}\widetilde{T}_{1,n}^{(2)}+\frac{1}{6}\widetilde{T}_{1,n}^{(3)}\widetilde{T}_{1,n}^{(4)})^{-1}$$

then we obtain

$$\|U_{1,n}^{\texttt{pre-cor}}(\tau_n)\xi-U_{1,n}(\tau_n)\xi\|_n\le C\tau_n^4||\xi||_n$$

which implies that the order of the scheme is $p=3$.

5. One dimensional linear transport equation with production. In this section we consider the system **(TP)**, when $\sigma\neq 0$ and $p(x)\neq 0$.

Here, we do not have at our disposition an explicit expression of the semigroup $U(t)$ as $U_0(t)$ or $U_1(t)$. Hence for representing $U(t)$ we will use the Dyson-Phillips formula:

$$V_0(t)=U_1(t),\qquad U(t):=\sum_{n=0}^{\infty}V_n(t),$$

where

$$V_{n+1}(t)=\int_0^t V_0(t-s)V_n(s)Ads,$$

$$[U_1(t)f](x,v)=\mathrm{e}^{-\int_0^t\sigma(x-sv)ds}f(\mathbf{x}-t\mathbf{v},\mathbf{v})\ \text{ and }\ [Af](x,v)=\frac{1}{2}\int_{-1}^{1}p(x)f(\mathbf{x},\mathbf{v}')d\mathbf{v}'.$$

Let us define an approximation of $U(t)$ by $U_N(t)$, where

$$U_N(t)f = [\sum_{k=0}^{N+1} V_k(t)]f = U_1(t)f + \int_0^t U_1(t-s)B(s)f ds,$$

with

$$\begin{aligned} B(s) =& U_1(s)A + \int_0^s U_1(s-s_1)U_1(s_1)A^2 ds_1 + \cdots \\ &+ \int_0^s \cdots \int_0^{s_N} U_1(s-s_1)U_1(s_1-s_2)\cdots U_1(s_N)A^{N+1} ds_N \cdots ds_1. \end{aligned} \tag{5.1}$$

Remark 5.1. The operator $U_N(t)$ is not himself a semigroup as $U_0(t)$ or $U_1(t)$, but it can act as the function $V(t)$ in the Chernoff's theorem. This will be shown in Appendix 1.

In the discrete version we denote by $W_{N,n}(n\tau_n)$ the operator which approximates $U_N(t)$ and is given by

$$[W_{N,n}(n\tau_n)\xi]_{i,j} = \sum_{k=0}^{N+1} [V_{k,n}(n\tau_n)\xi]_{i,j}$$

where $[V_{0,n}(n\tau_n)\xi]_{i,j} = [U_{1,n}(n\tau_n)\xi]_{i,j}$ is given in (4.3) and $V_{k,n}$ by the induction relation

$$[V_{k+1,n}(n\tau_n)\xi]_{i,j} = \tau_n \sum_{k=1}^{n} [V_{0,n}(n\tau_n - k\tau_n)V_{k,n}(k\tau_n)A_n\xi]_{i,j} \tag{5.2}$$

with

$$[A_n\xi]_{i,j} = \frac{1}{2}p_i k_n \sum_{l=-\mu_n}^{\mu_n-1} \xi_{i,l}.$$

which is independent of j. Since $U_{1,n}(n\tau_n)$ is a bounded operator in X_n, by a simple induction argument it follows from (5.2) that

$$\|V_{k,n}(n\tau_n)\xi\|_n = \mathcal{O}(\tau_n^k) \tag{5.3}$$

Theorem 5.2. *Under the assumption $2k_M < s_m$, we have the convergence of $W_{N,n}(t)$ to $U(t)$ in the sense of Kato.*

Proof. We have to prove that

$$\|W_{N,n}(t)P_n f - P_n U(t)f\|_n \to 0, \tag{5.4}$$

as $n \to \infty$.

First we prove that

$$W_{N,n}(k\tau_n)P_n f = P_n U_N(\tau_n)^k f. \tag{5.5}$$

The fact that $P_n \int_0^{\tau_n} U_1(\tau_n - s)V_{k-1}(s)Af(x,v)ds = \tau_n^k [U_{1,n}(\tau_n)A_n^k)\xi]_{i,j}$, shows

$$\begin{aligned} P_n U_N(\tau_n)f &= P_n[\sum_{k=0}^{N+1} V_k(\tau_n)f(x,v)] = [U_{1,n}(\tau_n) \sum_{k=0}^{N+1} \tau_n^k A_n^k \xi]_{i,j} \\ &= [U_{1,n}(\tau_n) \sum_{k=0}^{N+1} \tau_n^k A_n^k P_n f = W_{N,n}(\tau_n)P_n f. \end{aligned}$$

Hence, by taking $g = U_N(\tau_n)f$, we obtain

$$P_n U_N(\tau_n)^2 f = P_n U_N(\tau_n) g = W_{N,n}(\tau_n) P_n g = W_{N,n}(\tau_n)^2 P_n f,$$

and by induction we retrieve (5.5). Once the identity (5.5) is proven, we replace $W_{N,n}(t)P_n f$ by $P_n U_N(\tau_n)^n f$ in (5.4) and we use the isometric character of P_n (see Lemma 1.6), then we get

$$\|W_{N,n}(t)P_n f - P_n U(t) f\|_n = \|U_N(t/n)^n f - U(t) f\|.$$

Now, if $\omega = s_m - k_M$, thanks to Theorem 7.3, $U(t)$ satisfies $\|U(t)\| \le \mathrm{e}^{-\omega t}$, and since $2k_M < s_m$, we get $k_M < \omega$. So we can replace in Theorem 7.2, $S_0(t)$ by $U_1(t)$ and $B(s)$ by our operator defined in (5.1), and the Chernoff's Theorem (Theorem 7.1) proves that (5.4) holds. □

REMARK 5.3. *Since the numerical computation of $[W_N(\tau_n)\xi]_{i,j}$ is too complicated we restricted ourself to the standard schemes and thank to the above Theorem we make our comparison with $W_N(\tau_n)$.*

5.1. Euler's explicit and implicit schemes for linear transport equation with production. In the sequel we will use also the following operators Σ_n defined by

$$[\Sigma_n \xi]_{i,j,k} = \tau_n \sigma_i \xi_{j,k} \tag{5.6}$$

and

$$[\mathscr{A}_n \xi]_{i,j} = \frac{1}{2} p_i k_n \sum_{l=-\mu_n}^{\mu_n - 1} \xi_{j,l}, \tag{5.7}$$

for $(i,j) \in \mathcal{N}$ and $-\mu_n \le k \le \mu_n$. We remark that according to convention of Remark 3.1, in (5.7) j can take any values out of rang of x.

For these scheme we define two matrix operators $T_{2,n}^{\mathtt{Euler-exp}}$ and $T_{2,n}^{\mathtt{Euler-imp}}$ in $B(X_n)$ by

$$[T_{2,n}^{\mathtt{Euler-exp}}\xi]_{i,j} = (\xi_{i-j,j} - \xi_{i,j}) - [\Sigma_n\xi]_{i-j,i-j,j} + [\mathscr{A}_n\xi]_{i-j,i-j}\,,$$

if $(i,j) \in \mathcal{N}$ and

$$[T_{2,n}^{\mathtt{Euler-imp}}\xi]_{i,j} = (\xi_{i,j} - \xi_{i+j,j}) - \tau_n\sigma_i\xi_{i+j,j} + \frac{1}{2} p_i \tau_n k_n \sum_{l=-\mu_n}^{\mu_n-1} \xi_{i+l,l},$$

if $(i,j) \in \mathcal{N}$. For explicit Euler's scheme the approximated solution would be

$$[U_{2,n}^{\mathtt{Euler-exp}}\xi]_{i,j} := [\xi + T_{2,n}^{(1)}\xi]_{i,j} = (1 - \tau_n\sigma_{i-j})\xi_{i-j,j} + \frac{1}{2} p_{i-j} h_n \sum_{l=-p_n}^{p_n} \xi_{i-j,l}.$$

Our aim for explicit and implicit Euler's schemes is to get the order $p = 1$. So according to (5.3) for this scheme all the terms $V_{k,n}, k \ge 2$ can be neglected and it is enough to take into account $V_{0,n}$ and $V_{1,n}$, in other words, make a comparison only with $W_{0,n}$ which leads the following estimation:

$$\begin{aligned} |[W_{0,n}(\tau_n)\xi - U_{2,n}^{\mathtt{Euler-exp}}\xi]_{i,j}| = {} & |(\exp(-\tau_n\sigma_{i-j}) - 1 + \tau_n\sigma_{i-j})\xi_{i-j,j} \\ & + \frac{1}{2} p_{i-j}\tau_n k_n \sum_{l=-\mu_n}^{\mu_n-1} \xi_{i-j,l}(\exp(-\tau_n\sigma_{i-j}) - 1)|. \end{aligned}$$

Consequently

$$\|W_{0,n}(\tau_n)\xi - \eta\|_n \le A\tau_n^2\|\xi\|_n$$

where the constant A depends only on σ, but independent of τ_n.

For implicit Euler's scheme the approximated solution would be

$$[U_{2,n}^{\texttt{Euler-imp}}\xi]_{i,j} = [\left(I - T_{2,n}^{\texttt{Euler-imp}}\right)^{-1}\xi]_{i,j}$$

or

$$\begin{aligned}\xi_{i,j} = [\eta - T_{2,n}^{(2)}\eta]_{i,j} &= \eta_{i,j} - (\eta_{i,j} - \eta_{i+j,j}) - \tau_n\sigma_i\eta_{i+j,j} + \frac{1}{2}p_i\tau_n k_n \sum_{l=-\mu_n}^{\mu_n-1}\eta_{i+l,l} \\ &= \eta_{i+j,j} + \tau_n\sigma_i\eta_{i+j,j} - \frac{1}{2}p_i\tau_n k_n \sum_{l=-\mu_n}^{\mu_n-1}\eta_{i+l,l} = [(I+S)\eta]_{i+j,j}\end{aligned}$$

where $[S\eta]_{i,j} = \tau_n\sigma_{i-j}\eta_{i,j} - \frac{1}{2}p_{i-j}\tau_n k_n \sum_{l=-\mu_n}^{\mu_n-1}\eta_{i-j+l,l}$. So, we get $\xi_{i-j,j} = [(I+S)\eta]_{i,j}$ which gives,

$$[\eta]_{i,j} = [(I+S)^{-1}N\xi]_{i,j} \quad \text{where} \quad [N\xi]_{i,j} = \xi_{i-j,j}.$$

Therefore

$$\begin{aligned}[\eta]_{i,j} &= [(I-S)N\xi]_{i,j} + \mathcal{O}(\tau_n^2)[N\xi]_{i,j} \\ &= \xi_{i-j,j} - \tau_n\sigma_{i-j}\xi_{i-j,j} - \frac{1}{2}p_{i-j}\tau_n k_n \sum_{l=-\mu_n}^{\mu_n-1}[N\xi]_{i-j+l,l} + \mathcal{O}(\tau_n^2)\xi_{i-j,j} \\ &= (1-\tau_n\sigma_{i-j})\xi_{i-j,j} + \frac{1}{2}p_{i-j}\tau_n k_n \sum_{l=-\mu_n}^{\mu_n-1}\xi_{i-j,l} + \mathcal{O}(\tau_n^2)\xi_{i-j,j}\end{aligned}$$

Once more a comparison with respect to the our approximate solution $W_{0,n}(t)$ leads the following estimation:

$$\begin{aligned}|[W_{0,n}(\tau_n)\xi - U_{2,n}^{\texttt{Euler-imp}}\xi]_{i,j}| = |(\exp(-\tau_n\sigma_{i-j}) - 1 + \tau_n\sigma_{i-j})\xi_{i-j,j} + \frac{1}{2}p_{i-j}\tau_n k_n \\ \sum_{l=-\mu_n}^{\mu_n-1}\xi_{i-j,l}(\exp(-\tau_n\sigma_{i-j}) - 1)| + \mathcal{O}(\tau_n^2)\xi_{i-j,j}\end{aligned}$$

and

$$\|W_{0,n}(\tau_n)\xi - \eta\|_n = \mathcal{O}(\tau_n^2)$$

which gives the desired result.

5.2. Crank-Nicholson scheme for linear transport equation with production. For this scheme, we define two matrix operators $T_{2,n}^{(1)}$ and $T_{2,n}^{(2)}$ in $\mathcal{L}(X_n)$ as

$$[T_{2,n}^{(1)}\xi]_{i,j} := 2(\xi_{i-\frac{j}{2},j} - \xi_{i,j}) - \tau_n \sigma_{i-\frac{j}{2}} \xi_{i-\frac{j}{2},j} + [\mathscr{A}_n \xi]_{i-\frac{j}{2},i-\frac{j}{2}}$$

and

$$[T_{2,n}^{(2)}\xi]_{i,j} := 2(\xi_{i,j} - \xi_{i+\frac{j}{2},j}) - \tau_n \sigma_{i+\frac{j}{2}} \xi_{i+\frac{j}{2},j} + [\mathscr{A}_n \xi]_{i+\frac{j}{2},i+\frac{j}{2}}$$

We define also the following operator

$$[M_1\xi]_{i,j} = [(I - \frac{1}{2}T_1)(I + \frac{1}{2}T_1)^{-1}\xi]_{i,j} \quad \text{where} \quad [T_1\xi]_{i,j} = \tau_n \sigma_i \xi_{i,j} - [\mathscr{A}_n \xi]_{i,i}$$

We remark that

$$\begin{aligned}
[(I + \frac{1}{2}T_{2,n}^{(1)})\xi]_{i,j} &= \xi_{i,j} + (\xi_{i-\frac{j}{2},j} - \xi_{i,j}) - \frac{\tau_n \sigma_{i-\frac{j}{2}}}{2} \xi_{i-\frac{j}{2},j} + \frac{1}{2}[\mathscr{A}_n \xi]_{i-\frac{j}{2},i-\frac{j}{2}} \\
&= [(I - \frac{1}{2}T_1)\xi]_{i-\frac{j}{2},j}
\end{aligned}$$

and

$$\begin{aligned}
[(I - \frac{1}{2}T_{2,n}^{(2)})M_1\xi]_{i-j,j} &= [M_1\xi]_{i-j,j} - ([M_1\xi]_{i-j,j} - [M_1\xi]_{i-\frac{j}{2},j}) \\
&\quad + \frac{\sigma_{i-\frac{j}{2}}\tau_n}{2}[M_1\xi]_{i-\frac{j}{2},j} - \frac{p_{i-\frac{j}{2}}\tau_n}{4} k_n \sum_{l=-\mu_n}^{\mu_n - 1} [M_1\xi]_{i-\frac{j}{2},l} \\
&= [(I + \frac{1}{2}T_1)M_1\xi]_{i-\frac{j}{2},j} = [(I + \frac{1}{2}T_1)(I - \frac{1}{2}T_1)(I + \frac{1}{2}T_1)^{-1}\xi]_{i-\frac{j}{2},j} \\
&= [(I - \frac{1}{2}T_1)\xi]_{i-\frac{j}{2},j} = [(I + \frac{1}{2}T_{2,n}^{(1)})\xi]_{i,j}
\end{aligned}$$

By defining the approximate solution as

$$\begin{aligned}
[\eta]_{i,j} = [U_{2,n}^{\texttt{Cr-Ni}}\xi]_{i,j} &:= [(1 + \frac{1}{2}T_{2,n}^{(1)})(1 - \frac{1}{2}T_{2,n}^{(2)})^{-1}\xi]_{i,j} = [M_1\xi]_{i-j,j} \\
&= [(I - \frac{1}{2}T_1)(I + \frac{1}{2}T_1)^{-1}\xi]_{i-j,j} \\
&= [(I - \frac{1}{2}T_1)(I - \frac{1}{2}T_1 + \frac{1}{4}T_1^2 + \mathcal{O}(\tau_n^3))\xi]_{i-j,j} \\
&= [(I - T_1 + \frac{1}{2}T_1^2 + \mathcal{O}(\tau_n^3))\xi]_{i-j,j}
\end{aligned}$$

we get

$$\begin{aligned}
[\eta]_{i,j} &= \xi_{i-j,j} - \sigma_{i-j}\tau_n \xi_{i-j,j} + \frac{\sigma_{i-j}^2 \tau_n^2}{2}\xi_{i-j,j} + [\mathscr{A}_n\xi]_{i-j,i-j} - \sigma_{i-j}\tau_n [\mathscr{A}_n\xi]_{i-j,i-j} \\
&\quad + \Big[\frac{p_{i-j}\tau_n}{2}\mathscr{A}_n\xi\Big]_{i-j,i-j} + \mathcal{O}(\tau_n^3)[\xi]_{i-j,j} \\
&= (1 - \sigma_{i-j}\tau_n + \frac{\sigma_{i-j}^2\tau_n^2}{2})\xi_{i-j,j} + \frac{1}{2}p_{i-j}\tau_n k_n \sum_{l=-\mu_n}^{\mu_n-1} \xi_{i-j,l}(1 - \sigma_{i-j}\tau_n) \\
&\quad + \frac{p_{i-j}^2 \tau_n^2}{4} k_n \sum_{l=-\mu_n}^{\mu_n-1} \xi_{i-j,l} + \mathcal{O}(\tau_n^3)[\xi]_{i-j,j}
\end{aligned}$$

In this scheme any $V_{k,n}$, when $k \geq 3$ cannot affect on the order of rational approximation, so we shall make the comparison only with $W_{1,n}(t)$ which leads following estimation :

$$\begin{aligned}|[W_{1,n}(\tau_n)\xi - U_{2,n}^{\mathtt{Cr-Ni}}\xi]_{i,j}| &= |(\exp(\tau_n\sigma_{i-j}) - 1 + \tau_n\sigma_{i-j} + \frac{\sigma_{i-j}^2\tau_n^2}{2})\xi_{i-j,j} \\ &+ \frac{1}{2}p_{i-j}\tau_n k_n \sum_{l=-\mu_n}^{\mu_n-1} \xi_{i-j,l}(\exp(-\tau_n\sigma_{i-j}) - 1 + \tau_n\sigma_{i-j}) \\ &+ \frac{p_{i-j}^2\tau_n^2}{4} k_n \sum_{l=-\mu_n}^{\mu_n-1} \xi_{i-j,l}(2\exp(-\tau_n\sigma_{i-j}) - 1)| - \mathcal{O}(\tau_n^3))\xi_{i-j,j}.\end{aligned}$$

and

$$\|[W_{1,n}(\tau_n)\xi - U_{2,n}^{\mathtt{Cr-Ni}}\xi\|_n = \mathcal{O}(\tau_n^3),$$

consequently the order of the scheme would be $p = 2$.

5.3. Predictor-corrector scheme for linear transport equation with production. . Here we define four matrix operators

- $[\widetilde{T}_{2,n}^{(1)}\xi]_{i,j} := 3[T_{0,n}^{\mathtt{Euler-exp}}\xi]_{i,j} + [T_{0,n}^{\mathtt{Euler-exp}}\xi]_{i-j,j} - 3[T_{0,n}^{\mathtt{Euler-exp}}\xi]_{i-\frac{j}{2},j} - \frac{2}{3}[\Sigma_n\xi]_{i-j,i-j,j}$ $-\frac{1}{3}[\Sigma_n\xi]_{i-2j,i-2j,j} - [\Sigma_n\xi]_{i-\frac{j}{2},i-\frac{j}{2},j} + [\Sigma_n\xi]_{i-\frac{3j}{2},i-\frac{3j}{2},j} + \frac{2}{3}[\mathscr{A}_n\xi]_{i-j,i-j} + \frac{1}{3}[\mathscr{A}_n\xi]_{i-2j,i-2j}$ $+ [\mathscr{A}_n\xi]_{i-\frac{j}{2},i-\frac{j}{2}} - [\mathscr{A}_n\xi]_{i-\frac{3j}{2},i-\frac{3j}{2}}$;
- $[\widetilde{T}_{2,n}^{(2)}\xi]_{i,j} := [\widetilde{T}_{0,n}^{(2)}\xi]_{i,j} + \frac{1}{2}[\Sigma_n\xi]_{i+\frac{j}{2},i+j,j} - \frac{3}{4}[\Sigma_n\xi]_{i-j,i-j,j} - \frac{3}{4}[\Sigma_n\xi]_{i,i,j} - \frac{1}{2}[\Sigma_n\xi]_{i-\frac{j}{2},i,j}$ $+\frac{1}{3}[\Sigma_n\xi]_{i,i+\frac{j}{2},j} - \frac{3}{2}[\Sigma_n\xi]_{i+\frac{j}{2},i+\frac{j}{2},j} + \frac{1}{6}[\Sigma_n\xi]_{i-j,i-\frac{j}{2},j} + \frac{3}{2}[\Sigma_n\xi]_{i-\frac{j}{2},i-\frac{j}{2},j} + \frac{3}{4}[\mathscr{A}_n\xi]_{i,i}$ $+\frac{1}{2}[\mathscr{A}_n\xi]_{i-\frac{j}{2},i} + \frac{3}{4}[\mathscr{A}_n\xi]_{i-j,i-j} + \frac{3}{2}[\mathscr{A}_n\xi]_{i+\frac{j}{2},i+\frac{j}{2}} - \frac{1}{3}[\mathscr{A}_n\xi]_{i,i+\frac{j}{2}} - \frac{3}{2}[\mathscr{A}_n\xi]_{i-\frac{j}{2},i-\frac{j}{2}}$ $-\frac{1}{6}[\mathscr{A}_n\xi]_{i-j,i-\frac{j}{2}} - \frac{1}{2}[\mathscr{A}_n\xi]_{i+\frac{j}{2},i+j}$;
- $[\widetilde{T}_{2,n}^{(3)}\xi]_{i,j} := [\widetilde{T}_{0,n}^{(3)}\xi]_{i,j} - \frac{2}{3}[\Sigma_n\xi]_{i,i+\frac{j}{2},j} - \frac{1}{3}[\Sigma_n\xi]_{i-j,i-\frac{j}{2},j} - [\Sigma_n\xi]_{i+\frac{j}{2},i+j,j} +$ $[\Sigma_n\xi]_{i-\frac{j}{2},i,j}$
$+\frac{2}{3}[\mathscr{A}_n\xi]_{i,i+\frac{j}{2}} + \frac{1}{3}[\mathscr{A}_n\xi]_{i-j,i-\frac{j}{2}} + [\mathscr{A}_n\xi]_{i+\frac{j}{2},i+j} - [\mathscr{A}_n\xi]_{i-\frac{j}{2},i}$;
- $[\widetilde{T}_{2,n}^{(4)}\xi]_{i,j} = 2(\xi_{i-\frac{j}{2},j} - \xi_{i,j}) - [\Sigma_n\xi]_{i-\frac{j}{2},i-\frac{j}{2},j} + [\mathscr{A}_n\xi]_{i-\frac{j}{2},i-\frac{j}{2}}$.

And we define also the following operator

$$[M_2\xi]_{i,j} = [(I - \frac{1}{3}T_1)(I + \frac{2}{3}T_1 + \frac{1}{6}T_1^2)^{-1}\xi]_{i,j} \quad \text{where} \quad [T_1\xi]_{i,j} = [\Sigma_n\xi]_{i,i,j} - [\mathscr{A}_n\xi]_{i,i}.$$

By a calculation one can prove that †

$$\begin{aligned}[(I + \frac{1}{3}\widetilde{T}_{2,n}^{(1)})\xi]_{i,j} &= \frac{2}{3}(I - \frac{1}{3}T_1)\xi_{i-j,j} + \frac{1}{3}(I - \frac{1}{3}T_1)\xi_{i-2j,j} + (I - \frac{1}{3}T_1)\xi_{i-\frac{j}{2},j} \\ &- (I - \frac{1}{3}T_1)\xi_{i-\frac{3}{2}j,j}\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}[(I - \frac{2}{3}\widetilde{T}_{2,n}^{(2)} + \frac{1}{6}\widetilde{T}_{2,n}^{(3)}\widetilde{T}_{2,n}^{(4)})M_2\xi]_{i-j,j} &= \frac{2}{3}[(I - \frac{1}{3}T_1)\ \xi]_{i-j,j} + \frac{1}{3}[(I - \frac{1}{3}T_1)\xi]_{i-2j,j} \\ &+ [(I - \frac{1}{3}T_1)\xi]_{i-\frac{j}{2},j} - [(I - \frac{1}{3}T_1)\xi]_{i-\frac{3}{2}j,j}.\end{aligned} \tag{5.9}$$

† This is a long calculation which can be find in an extended version of this paper located in the second author's webpage `http://www-math.univ-poitiers.fr/~emamirad/`

Therefore, we have

$$[(I+\frac{1}{3}\widetilde{T}^{(1)}_{2,n})\xi]_{i,j} = [(I-\frac{2}{3}\widetilde{T}^{(2)}_{2,n}+\frac{1}{6}\widetilde{T}^{(3)}_{2,n}T^{(4)}_{2,n})M_2\xi]_{i-j,j}$$

and consequently, by defining

$$\begin{aligned}[\eta]_{i,j} = [U^{\texttt{pre-cor}}_{2,n}\xi]_{i,j} := \ & [(1-\frac{2}{3}T^{(2)}_{2,n}+\frac{1}{6}T^{(3)}_{2,n}T^{(4)}_{2,n})^{-1}(1+\frac{1}{3}T^{(1)}_{2,n})\xi]_{i,j} = [M_2\xi]_{i-j,j} \\ = \ & [(I-\frac{1}{3}T_1)(I+\frac{2}{3}T_1+\frac{1}{6}T_1^2)^{-1}\xi]_{i-j,j} \\ = \ & [(I-T_1+\frac{1}{2}T_1^2-\frac{1}{6}T_1^3+\mathcal{O}(\tau_n^4))\xi]_{i-j,j} \\ = \ & (1-\sigma_{i-j}\tau_n+\frac{\sigma^2_{i-j}\tau_n^2}{2}-\frac{\sigma^3_{i-j}\tau_n^3}{6})\xi_{i-j,j}+\Big(1-\sigma_{i-j}\tau_n+\frac{\sigma^2_{i-j}\tau_n^2}{2} \\ & +\frac{1}{2}p_{i-j}(1-\sigma_{i-j}\tau_n+\frac{1}{3}p_{i-j}\tau_n^2)\Big)[\mathscr{A}_n\xi]_{i-j,i-j}+\mathcal{O}(\tau_n^4))\xi_{i-j,j}\end{aligned}$$

In this scheme any $V_{k,n}$, when $k \geq 4$ cannot affect on the order of rational approximation, so we shall take $W_{2,n}(t)$ as the approximate solution and a comparison with respect to this approximate solution leads to following estimation :

$$\begin{aligned}|[W_{2,n}(\tau_n)\xi-\eta]_{i,j}| = \ & |(\exp(-\tau_n\sigma_{i-j})-1+\tau_n\sigma_{i-j}-\frac{\sigma^2_{i-j}\tau_n^2}{2}+\frac{\sigma^3_{i-j}\tau_n^3}{6})\xi_{i-j,j} \\ & +\frac{1}{2}p_{i-j}\tau_n k_n\sum_{l=-\mu_n}^{\mu_n-1}\xi_{i-j,l}(\exp(-\tau_n\sigma_{i-j})-1+\tau_n\sigma_{i-j}-\frac{\sigma^2_{i-j}\tau_n^2}{2}) \\ & +\frac{1}{4}p^2_{i-j}\tau_n^2 k_n\sum_{l=-\mu_n}^{\mu_n-1}\xi_{i-j,l}(2\exp(-\tau_n\sigma_{i-j})-1+\tau_n\sigma_{i-j}) \\ & +\frac{1}{12}p^3_{i-j}\tau_n^3 k_n\sum_{l=-\mu_n}^{\mu_n-1}\xi_{i-j,l}(6\exp(-\tau_n\sigma_{i-j})-1)-\theta(\tau_n^4))\xi_{i-j,j}|\end{aligned}$$

and

$$\|[W_{2,n}(\tau_n)\xi-U^{\texttt{pre-cor}}_{2,n}\xi\|_n = \mathcal{O}(\tau_n^4),$$

consequently the order of the scheme would be $p=3$.

6. The numerical illustrations . This section is devoted to give the numerical illustrations for Euler explicit and implicit, Crank-Nicolson and Predictor-Corrector methods. So, we use the Fortran 77 compiler to give the numerical approximation in different cases of our transport equation. The numerical simulations which realized for a positive function and with non entrance boundary condition give an idea on the distribution of particles in the phases space and verify also our theoretical results in this work.

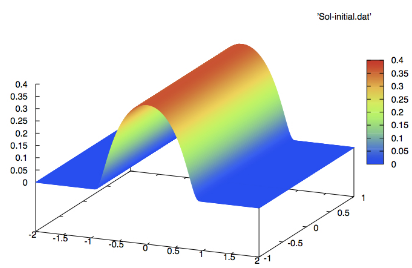
In the sequel, we will give some numerical examples for our different schemes. In those examples, we look to the evolution of the transport equation in five times. For $a=1, m_n=200$ and $\mu_n=100$n or, in order to get $\tau_n k_n/h_n=1$ we have to take $\tau_n=0.5$ which fix the choice of n. The five times which will be illustrated there are:

$t = k\tau_n$ for $k = 1, 3$ and $k = 400$. Also, for those examples we will take the following initial data

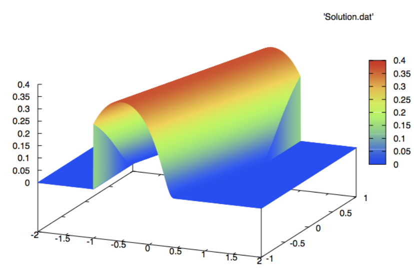
$$f(x, v) = \exp\left(\frac{-1}{1 - x^2}\right) \tag{6.1}$$

which is independent of the velocity v.

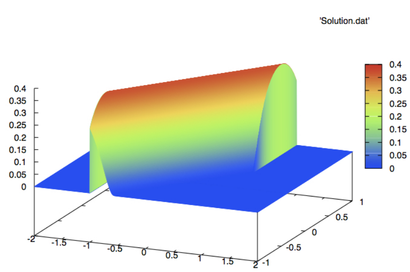
6.1. The numerical approach in the case of collision-free transport. To have an idea on the evolution of particles in the case of collision-free transport problem we have compiled the approximation of the exact solution given by (3.2), since as we have proved in Theorem 3.3 all the different schemes ends to this discrete form of the exact solution. In the following figures we illustrate numerically the evolution of the pure translation of an initial solution with non entrance boundary condition. We remark that the large time, which corresponds here to $k = 400$, it remains always a residual that corresponds to $f(x, 0)$ of the initial data, since we have not excluded zero from the velocity interval $[-1, 1]$.



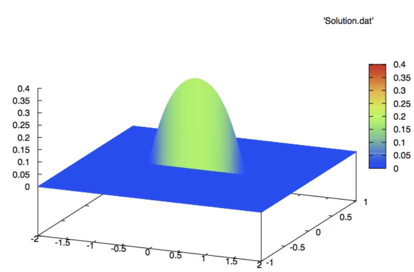
`Initial solution`



`Solution for` $k = 1$



`Solution for` $k = 3$



`Solution for` $k = 400$

6.2. Error estimates. In the case of transport with pure absorption, we find the same feature of numerical illustration. Since in this case we have an explicit representation of the solution given in (4.1) it is more interesting to give the error made by the schemes of Euler, Crank-Nicolson and Predictor-Corrector.

In fact, if $\epsilon_n = \|U_{1,n}(k\tau_n)P_n f - P_n U_1(k\tau_n) f\|_n$, then for the function f given in (6.1) and $k = 1$ we have

Euler's implicit	Euler's explicit	Crank-Nicolson	Predictor-Corrector
$\epsilon_n < 4.5 \times 10^{-4}$	$\epsilon_n < 4.5 \times 10^{-4}$	$\epsilon_n < 4 \times 10^{-6}$	$\epsilon_n < 3 \times 10^{-8}$

In the case of transport problem with production term, we do not have an explicit solution at our disposal, so we compute $\varepsilon_n = \|W_{N,n}(k\tau_n)\xi - \eta\|_n$, then for $N = k = 1$ and we get the following table for correspondent η.

Euler's implicit	Euler's explicit	Crank-Nicolson	Predictor-Corrector
$\varepsilon_n < 4.5 \times 10^{-4}$	$\varepsilon_n < 4.5 \times 10^{-4}$	$\varepsilon_n < 7 \times 10^{-6}$	$\varepsilon_n < 3 \times 10^{-7}$

7. Appendix. The well-known Chernoff's Theorem asserts that

THEOREM 7.1. *If X is a Banach space and $\{V(t)\}_{t\geq 0}$ is a family of contractions on X with $V(0) = I$. Suppose that the derivative $V'(0)f$ exists for all f in a set $\mathcal{D}$ and the closure Λ of $V'(0)\mid_{\mathcal{D}}$ generates a C_0–semigroup $S(t)$ of contractions. Then, for each $f \in X$,*

$$\lim_{n\to\infty} \|V(\frac{t}{n})^n f - S(t)f\| = 0, \tag{7.1}$$

uniformly for t in compact subsets of $\mathbb{R}_+$.

In this section we will use the Chernoff's theorem to prove the following result.

THEOREM 7.2. *Let A be the generator of a C_0-semigroup $S_0(t)$ such that $\|S_0(t)\| \leq \mathrm{e}^{-\omega t}$ $(\omega \geq 0)$, and $B(t)$ be a family of bounded operators such that $\|B(t)\| < \omega$ for all $t \geq 0$, and $A + B(0)$ defined in the $D(A)$ generates a C_0-semigroup $S(t)$ of contractions. Then, the conclusion of (7.1) holds for $V(t) := S_0(t) + \int_0^t S_0(t-s)B(s)ds$.*

Proof. We remark that $V(0) = I$, $V'(0)f = (A + B(0))f$ for all $f \in D(A)$ and finally $V(t)$ is of contraction. In fact,

$$\begin{aligned}\|V(t)\| &\leq \|S_0(t)\| + \|\int_0^t S_0(t-s)B(s)ds\| \\ &\leq \mathrm{e}^{-\omega t} + b\int_0^t \mathrm{e}^{-\omega(t-s)}ds = \left(1 - \frac{b}{\omega}\right)\mathrm{e}^{-\omega t} + \frac{b}{\omega} \leq 1,\end{aligned}$$

where $b = \sup_{t\geq 0} \|B(t)\|$. Since all the assumptions of Theorem 7.1 are fulfilled, the conclusion infers from this Theorem. □

In [5], we have proved a similar version of this theorem where $V(t) := S_0(t) + \int_0^t S_0(s)B(0)ds$ and we have proved also the following theorem:

THEOREM 7.3. *In the Banach space $X = L^1((-a,a)\times[-1,1])$ let us define the operators $T_0 f := -v\partial f/\partial x$, $T_1 f := T_0 f - \sigma(x)f$, $\widetilde{T}f := T_0 f + Af$ and $Tf := T_1 f + Af$ (A being defined in Remark 1.5). Any of these operators defined on $D(T_0) := \{f \in X\ :\ v\partial f/\partial x \in X,\ f(-a, v \geq 0) = 0$ and $f(a, v \leq 0) = 0\}$ generates a C_0-semigroup which is given respectively by:*

(0) *$U_0(t)$ which are contractions;*
(1) *$U_1(t)$ with $\|U_1(t)\| \leq \mathrm{e}^{-s_m t}$;*
(2) *$V(t)$ with $\|V(t)\| \leq \mathrm{e}^{k_M t}$;*
(3) *$U(t)$ with $\|U(t)\| \leq \mathrm{e}^{(k_M - s_m)t}$.*

This Theorem is already used in the proof of Theorem 5.2.

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