Kinetic energy control in the MAC discretization of the compressible Navier-Stokes equations

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Abstract

We present in this short note a simple construction of the convection operator in the variable density Navier-Stokes equations (*i.e.* the discrete analogue of $\partial_t(\rho \boldsymbol{u}) + \mathrm{div}(\rho \boldsymbol{u} \times \boldsymbol{u})$, where ρ stands for the density and $\boldsymbol{u}$ for the velocity) for MAC discretizations, which ensures the control of the kinetic energy. We thereby extend a similar construction performed in previous works for staggered finite element non-conforming discretizations, and, consequently, the stability results which were obtained for implicit or pressure correction schemes for compressible barotropic flows.

Key words : Compressible Navier-Stokes equations, staggered discretizations, MAC scheme.

1 Introduction

Let Ω be a domain of $\mathbb{R}^d$, $d = 2$ or $d = 3$, suitable for a discretization with the MAC scheme (*i.e.* with boundaries parallel to the hyperplanes spanned by $d-1$ vectors of the canonical basis of $\mathbb{R}^d$), and let ρ and $\boldsymbol{u}$ be regular density and velocity fields defined on Ω and satisfying the mass balance:

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0.$$

Let z be a regular scalar function defined over Ω. For simplicity, assume that the velocity $\boldsymbol{u}$ vanishes at the boundary of Ω; then, the following identity holds:

$$\int_\Omega \big[\partial_t(\rho z) + \mathrm{div}(\rho z \boldsymbol{u})\big]\, z \,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\,\frac{d}{dt}\int_\Omega \rho z^2 \,\mathrm{d}\boldsymbol{x}. \tag{1}$$

If z is the i^{th} component of $\boldsymbol{u}$, $\partial_t(\rho z) + \mathrm{div}(\rho z \boldsymbol{u})$ is the convection part of the momentum balance equation projected along the i^{th} vector of the canonical basis of

$\mathbb{R}^d$; applying identity (1) to each component of $\boldsymbol{u}$ thus yields the central argument of the proof of the kinetic energy theorem.

The aim of this short note is to propose a discrete convection operator for the MAC scheme (see [8] for the seminal paper, [6, 7] for the first implementations for compressible flows and [10] for a review) which satisfies a discrete analogue of (1). The result given here is the only ingredient necessary to extend to the MAC discretization the stability studies already performed for staggered discretizations based on low-order nonconforming finite elements (namely the Crouzeix-Raviart or Rannacher-Turek elements) for incompressible [2] or barotropic compressible [4, 5] flows.

This note is organized as follows: we first give an abstract discrete analogue to identity (1), and then show how to apply it to a MAC discretization.

2 An abstract discrete stability result

The proof of the stability result stated in this section may be found in [4], and is valid for a quite general mesh. Let Ω be split in control volumes $\bar\Omega = \cup_{K\in\mathcal{M}} \bar K$, and let us denote the internal face of the mesh separating the cells K and L by $\sigma = K|L$. We suppose given two families of positive real numbers $(\rho^*_K)_{K\in\mathcal{M}}$ and $(\rho_K)_{K\in\mathcal{M}}$ satisfying the following set of equations:

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}\ (\rho_K - \rho^*_K) + \sum_{\sigma=K|L} F_{\sigma,K} = 0, \tag{2}$$

where $F_{\sigma,K}$ is a quantity associated to the edge σ and to the control volume K; we suppose that, for any internal edge $\sigma = K|L$, $F_{\sigma,K} = -F_{\sigma,L}$. The relation (2) may be seen as a discrete mass balance, namely the finite volume discretization of a balance equation of the form $\partial_t\rho + \operatorname{div}(\rho\boldsymbol{u}) = 0$; in this case, ρ^*_K and ρ_K stand for the approximation of the density ρ over K respectively at the beginning and end of the time step δt, $|K|$ is the measure of K and $F_{\sigma,K}$ is the mass flux coming out of K through σ, *i.e.* an approximation of the integral over σ of the quantity $\rho\boldsymbol{u}\cdot\boldsymbol{n}_{K,\sigma}$ where $\boldsymbol{n}_{K,\sigma}$ stands for the normal vector to σ outward K. Note that the sum of the fluxes is restricted to the internal edges of the mesh, which implicitly reflects the fact that the normal velocity is supposed to be zero at the boundary.

Let $(z^*_K)_{K\in\mathcal{M}}$ and $(z_K)_{K\in\mathcal{M}}$ be two families of real numbers. For any internal edge $\sigma = K|L$, we define z_σ by $z_\sigma = (z_K + z_L)/2$, and the convection operator $C_\mathcal{M}$ by:

$$\forall K \in \mathcal{M}, \quad (C_\mathcal{M} z)_K = \frac{1}{\delta t}\ (\rho_K\, z_K - \rho^*_K\, z^*_K) + \frac{1}{|K|} \sum_{\sigma=K|L} F_{\sigma,K}\, z_\sigma.$$

The quantity $(C_\mathcal{M}\, z)_K$ may be seen as a finite volume approximation over K of $\partial_t(\rho z) + \operatorname{div}(\rho\boldsymbol{u}z)$, where z is a generic scalar function. Then the following stability property holds:

$$\sum_{K\in\mathcal{M}} |K|\ z_K\ (C_\mathcal{M}\, z)_K \geq \frac{1}{2} \sum_{K\in\mathcal{M}} \frac{|K|}{\delta t}\ \left[\rho_K\, z_K^2 - \rho^*_K\, {z^*_K}^2\right]. \tag{3}$$

This relation is a discrete analogue of (1), which is thus satisfied provided that the convection operator $C_{\mathcal{M}}$ satisfies a consistency property with the discrete mass balance, namely (2), which can be reformulated by saying that $(C_{\mathcal{M}}\, z)_K = 0$, for any $K \in \mathcal{M}$ and for any constant discrete function z (*i.e.*, without loss of generality, if $z_K = 1$, for any $K \in \mathcal{M}$).

3 A MAC convection operator with kinetic energy control

We now need to specialize the presentation to the MAC discretization. We suppose that $d = 2$ for the sake of simplicity and use the notations given on Figure 1.

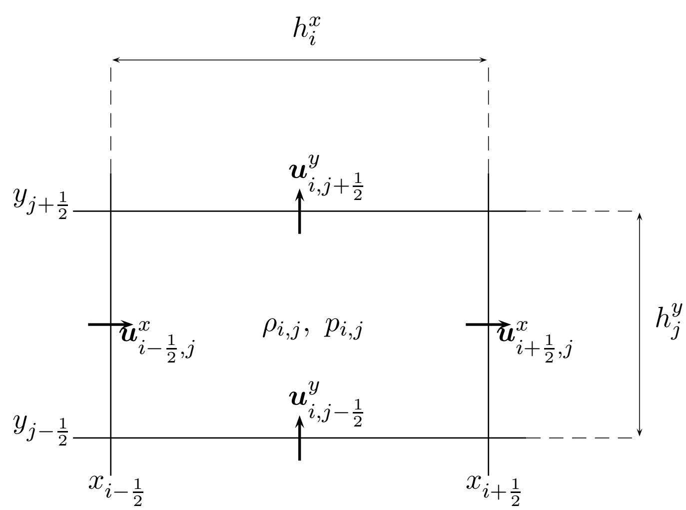

Figure 1: Mesh and unknowns.

On the cell $K_{i,j} = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$, the mass balance obtained by a backward Euler discretization reads:

$$\frac{|K_{i,j}|}{\delta t}\,(\rho_{i,j} - \rho^*_{i,j}) + F^x_{i+\frac{1}{2},j} + F^y_{i,j+\frac{1}{2}} - F^x_{i-\frac{1}{2},j} - F^y_{i,j-\frac{1}{2}} = 0, \qquad (4)$$

with, for instance, $F^x_{i+\frac{1}{2},j} = h^y_j\ \boldsymbol{u}^x_{i+\frac{1}{2},j}\ \tilde\rho_{i+\frac{1}{2},j}$, where $\tilde\rho_{i+\frac{1}{2},j}$ stands for an approximation of the density at the face, which, for the aim pursued here, may be obtained by any reasonable interpolation formula.

Let us now turn to the discretization of the momentum balance equation. With the notations of Figure 2, the equation for the x-component of $\boldsymbol{u}$ posed over the control volume $K^x_{i-\frac{1}{2},j}$ reads, with a backward Euler discretization:

$$\frac{|K^x_{i-\frac{1}{2},j}|}{\delta t}\Big[\rho_{i-\frac{1}{2},j}\, \boldsymbol{u}^x_{i-\frac{1}{2},j} - \rho^*_{i-\frac{1}{2},j}\, (\boldsymbol{u}^x_{i-\frac{1}{2},j})^*\Big] + F^x_{i,j}\ \boldsymbol{u}^x_{i,j} + F^y_{i-\frac{1}{2},j+\frac{1}{2}}\ \boldsymbol{u}^x_{i-\frac{1}{2},j+\frac{1}{2}}$$
$$- F^x_{i-1,j}\ \boldsymbol{u}^x_{i-1,j} - F^y_{i-\frac{1}{2},j-\frac{1}{2}}\ \boldsymbol{u}^x_{i-\frac{1}{2},j-\frac{1}{2}} + (T_d)_{i-\frac{1}{2},j} + (\boldsymbol{\nabla} p)_{i-\frac{1}{2},j} = 0,$$

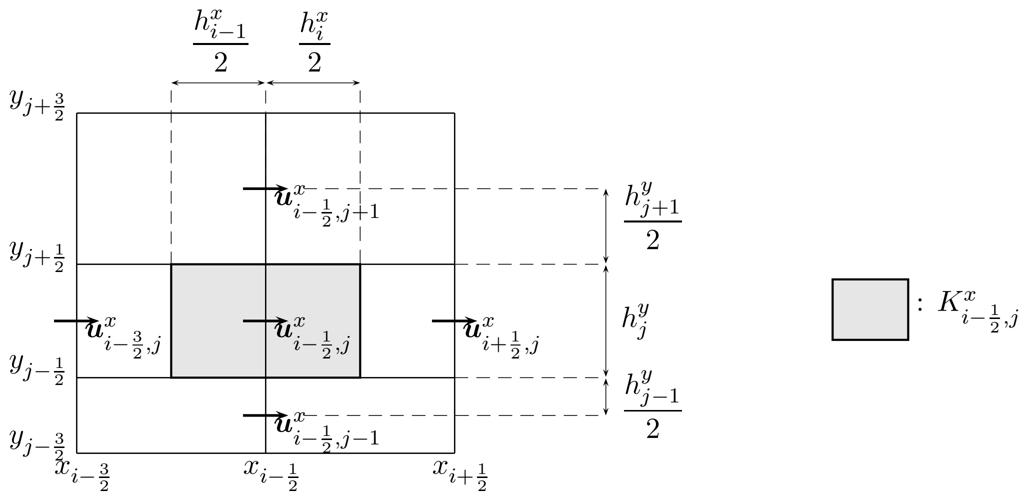

Figure 2: Notations for the control volume $K^x_{i-\frac{1}{2},j}$.

where $(T_d)_{i-\frac{1}{2},j}$ and $(\boldsymbol{\nabla} p)_{i-\frac{1}{2},j}$ stand for the approximation of the diffusion and pressure gradient term respectively, and $(\boldsymbol{u}^x_{i-\frac{1}{2},j})^*$ is the x-component of the velocity on $K^x_{i-\frac{1}{2},j}$ at the previous time step.

Our task is now to define the approximation of the densities $\rho_{i-\frac{1}{2},j}$ and $\rho^*_{i-\frac{1}{2},j}$, of the mass fluxes $F^x_{i,j}$, $F^y_{i-\frac{1}{2},j+\frac{1}{2}}$, $F^x_{i-1,j}$ and $F^y_{i-\frac{1}{2},j-\frac{1}{2}}$ and of the velocities $\boldsymbol{u}^x_{i,j}$, $\boldsymbol{u}^x_{i-\frac{1}{2},j+\frac{1}{2}}$, $\boldsymbol{u}^x_{i-1,j}$ and $\boldsymbol{u}^x_{i-\frac{1}{2},j-\frac{1}{2}}$ in such a way that the theory of Section 2 applies (and so, in particular, such that the resulting convection operator vanishes for any constant velocity field).

As in Section 2, the discretization of face velocities is given by a centered approximation:

$$\boldsymbol{u}^x_{i,j} = \frac{1}{2}\,(\boldsymbol{u}^x_{i-\frac{1}{2},j} + \boldsymbol{u}^x_{i+\frac{1}{2},j}), \qquad \boldsymbol{u}^x_{i-\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\,(\boldsymbol{u}^x_{i-\frac{1}{2},j} + \boldsymbol{u}^x_{i-\frac{1}{2},j+1}),$$
$$\boldsymbol{u}^x_{i-1,j} = \frac{1}{2}\,(\boldsymbol{u}^x_{i-\frac{3}{2},j} + \boldsymbol{u}^x_{i-\frac{1}{2},j}), \qquad \boldsymbol{u}^x_{i-\frac{1}{2},j-\frac{1}{2}} = \frac{1}{2}\,(\boldsymbol{u}^x_{i-\frac{1}{2},j} + \boldsymbol{u}^x_{i-\frac{1}{2},j-1}).$$

Let us now write the discrete mass balance for the mesh $K_{i-1,j}$ (that is (4) with $i-1$ instead of i), and sum with (4). We get:

$$\frac{1}{\delta t}\,\Big[(|K_{i-1,j}|\;\rho_{i-1,j} + |K_{i,j}|\;\rho_{i,j}) - (|K_{i-1,j}|\;\rho^*_{i-1,j} + |K_{i,j}|\;\rho^*_{i,j})\Big] + \Big[F^x_{i-\frac{1}{2},j}$$
$$+F^x_{i+\frac{1}{2},j}\Big] + \Big[F^y_{i-1,j+\frac{1}{2}} + F^y_{i,j+\frac{1}{2}}\Big] - \Big[F^x_{i-\frac{3}{2},j} + F^x_{i-\frac{1}{2},j}\Big] - \Big[F^y_{i-1,j-\frac{1}{2}} + F^y_{i,j-\frac{1}{2}}\Big] = 0.$$

In view of the result of Section 2, we see from this relation shows that we obtain the desired stability property taking for the densities:

$$|K^x_{i-\frac{1}{2},j}|\ \rho_{i-\frac{1}{2},j} = \frac{1}{2}\ \Big[|K_{i-1,j}|\ \rho_{i-1,j} + |K_{i,j}|\ \rho_{i,j}\Big],$$
$$|K^x_{i-\frac{1}{2},j}|\ \rho^*_{i-\frac{1}{2},j} = \frac{1}{2}\ \Big[|K_{i-1,j}|\ \rho^*_{i-1,j} + |K_{i,j}|\ \rho^*_{i,j}\Big],$$

and for the mass fluxes:

$$F^x_{i,j} = \frac{1}{2}\ \Big[F^x_{i-\frac{1}{2},j} + F^x_{i+\frac{1}{2},j}\Big] \qquad F^y_{i-\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\ \Big[F^y_{i-1,j+\frac{1}{2}} + F^y_{i,j+\frac{1}{2}}\Big]$$
$$F^x_{i-1,j} = \frac{1}{2}\ \Big[F^x_{i-\frac{3}{2},j} + F^x_{i-\frac{1}{2},j}\Big] \qquad F^y_{i-\frac{1}{2},j-\frac{1}{2}} = \frac{1}{2}\ \Big[F^y_{i-1,j-\frac{1}{2}} + F^y_{i,j-\frac{1}{2}}\Big]$$

This construction may be easily transposed to the y-component of the velocity, and, more generally, to the three-dimensional case.

Remark 1 Note that, rather surprisingly, depending on the discretization of the mass fluxes in the mass balance equation, the density in the cells $K_{i-1,j}$ or $K_{i+1,j}$ may appear in the expression of the $F^x_{i,j}$, even if this flux is associated to a face included in $K_{i,j}$.

Remark 2 (Boundary conditions) Usually, when using a MAC discretization, the Dirichlet boundary conditions for the velocity are directly prescribed to the boundary velocity degrees of freedom. In this case, the theory of Section 2 needs to be slightly adapted, because no balance equation is satisfied on the (half-) dual cells associated to these degrees of freedom. The proof that the stability inequality (3) still holds is given in [2, proof of Theorem 3.1]. Note however that, because of this particular feature of staggered discretizations, if the velocity is not prescribed to zero (but, let us say, to $\boldsymbol{u}_D$) at the boundary, the kinetic energy flux entering the domain is consistent with what may be anticipated from the boundary condition (*i.e.* $\frac{1}{2}\ \rho_D\ |\boldsymbol{u}_D|^2\ \boldsymbol{u}_D \cdot \boldsymbol{n}$, with ρ_D the density at the boundary and $\boldsymbol{n}$ the outward normal vector), but cannot be expressed as a function of the boundary data only [2, Remark 3].
When the velocity is not prescribed, an explicit expression of the kinetic energy flux across the boundary of the domain can be established, and this result may be used to derive a discrete artificial boundary condition able to cope with inward flows, purposely built to yield an energy estimate (see [3] for a work following similar ideas in the incompressible case). This artificial condition has been successfully tested to compute natural convection external flows [1].

Remark 3 (Upwinding of the velocity) If an upwind approximation is used for the velocity in the momentum flux through the faces of the dual cells, Inequality (3) is still satisfied, and an additional dissipation (non-negative) term appears at the right hand side [2, Remark 2].

Remark 4 (Forward Euler upwind scheme) The stability result (3) is proven to hold in [9] with a forward upwind discretization, under a CFL restriction of the time step. This allows to derive, without loss of stability at small time step, a semi-explicit version of any of the proposed implicit schemes.

Remark 5 (Using this construction in a pressure correction scheme) The implementation of the proposed construction of the convection operator is not straightforward in a pressure correction algorithm, since, usually, the mass balance equation is solved after the momentum balance equation. The choice of [2, 4, 5] to obtain stability is to start from (*i.e.* to use for Equation (4)) the mass balance at the previous time step. Denoting the time level by the superscript n, a linearly-implicit time discretization yields a semi-discrete convection term which reads $(\rho^n\, \boldsymbol{u}^{n+1} - \rho^{n-1}\, \boldsymbol{u}^n)/\delta t + \mathrm{div}(\rho^n \boldsymbol{u}^{n+1} \times \boldsymbol{u}^n)$.

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