Convergence and Sensitivity Analysis of Repair Algorithms in 1D

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Abstract

We prove the convergence of some repair algorithms for linear advection in dimension one. The convergence depends on the size of the box where the distribution of the mass excess is performed. Various numerical examples illustrate the theoretical results. Applications to gas dynamics in dimension one is also discussed.

Key words : Finite Volume Schemes, Repair algorithms, TVD schemes, Gas Dynamics.

1 Introduction

The aim of this work is the convergence analysis of some very simple Repair Algorithms. To our knowledge such a proof has never been given, even for simple problems. The Repair algorithms we consider are very much in the spirit of what was proposed by M. Shashkov and B. Wendroff in [11]. The underlying idea of Repair methods is related to the fact that conservative remapping methods (present in Arbitrary-Lagrangian-Eulerian (ALE) framework for example) may not necessarily preserve a maximum principle: *no new extrema should be generated during the remapping process*. If so a Repair method can remove these new maxima (resp. minima) by distributing (resp. taking) the amount of the associated conservative variable to (resp. from) the neighborhood.
In this paper we limit the theoretical study to transport equation in 1D and to some simple remapping strategy in 1D. However we hope that some of the conclusions of

the study can give insights even for multidimensional repair algorithms for systems of PDE's, such as compressible gas dynamics.
The outline of this paper is as follows. In the second section, we describe the repair algorithm for the linear transport. In section three we adapt the repair algorithm for the remapping process. The numerical results on 1D linear transport are gathered in the fourth section, whereas the fifth section presents some numerical results for 1D gas dynamics. Finally the conclusion and the plan for future developments is presented in the last section.

2 Repair algorithm for linear transport

Let us consider the equation of transport in 1D, with constant velocity

$$\partial_t u + a\partial_x u = 0, \quad a > 0. \tag{1}$$

The initial condition is $u(0,x) = u_0(x)$ where $u_0 \in L^1(\mathbf{R}) \cap BV(\mathbf{R})$. Let us consider a mesh, that is uniform even if our result extends to non uniform mesh. The mesh size is denoted by Δx. The time step is denoted by Δt. Thus u_j^n stands for the numerical solution in cell j at time step n.

The repair strategy amounts to: 1) compute a prediction of the numerical solution at time step $n+1$ using a "reasonable" and "local" scheme, as instance this scheme can be a high order non monotone scheme or a highly anti-dissipative scheme, 2) check if the new value satisfies a local maximum principle, 3) if the new value does not satisfy the local maximum principle, then repair it. Repairing means changing the value of the unknown for 3) to be fulfilled. A difficulty is that one wants the total mass to be preserved. So we need to describe in details how to redistribute the mass of the repaired quantity, such as the total mass is preserved.

This family of algorithms can be local if one redistributes the mass in a local box around the cell that needs to be repaired, or global if one redistributes the mass in the entire domain. It has been a debate since the early time of the Repair Algorithms to decide whether local repairing is better or not than global repairing. On one hand, global repairing is a more simple algorithm than local repairing. On the other hand our analysis implies that we can not prove the convergence of a global repair process. This is a theoretical indication that global repair process can be dangerous in some cases. Moreover a global repair process breaks the causality by instantaneously spreading mass all over the domain. The numerical experiments show it is indeed the case, in particular when the prediction step or the underlying physics is oscillating.

Let us describe in detail the repair algorithm that we analyze in this work.

2.1 The prediction scheme

First, one computes the new value of the unknown using the finite volume and conservative scheme

$$\boxed{\frac{\overline{u}_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n + c_{j+\frac{1}{2}}^n - u_{j-1}^n - c_{j-\frac{1}{2}}^n}{\Delta x} = 0,} \tag{2}$$

where u_j^n is the mean value of $u(x)$ in cell j: $[x_{x-\frac{1}{2}}; x_{x+\frac{1}{2}}]$.
The Courant or CFL number is less than one ($a > 0$)

$$\nu = a\frac{\Delta t}{\Delta x} \leq 1. \tag{3}$$

The scheme is *a priori* different from the upwind scheme. Thus all the difference between the upwind scheme and the scheme used is embedded in the definition of the correction flux $ac_{j+\frac{1}{2}}^n$ for all j. This correction flux can be either a linear or a non linear function of (u_j^n). The only assumption is

$$\exists C_1 > 0,\ \exists k \in \mathbf{N}, \quad |c_{j+\frac{1}{2}}^n| \leq C_1 \sum_{j-k\leq q\leq j+k} |u_q - u_{q-1}|. \tag{4}$$

The hypothesis (4) essentially means that the flux is defined as the upwind flux plus a correction. Of course the correction is zero if the numerical profile is flat (that is if $u_q - u_{q-1} \equiv 0$ in a neighborhood of cell j) : (4) is compatible with such a principle. The hypothesis is true for the Lax-Wendroff scheme as instance, moreover all non linear TVD algorithms satisfy (4).

2.2 Correction step

The spirit of this repair algorithm is to compare $\overline{u}_j^{n+1}$ with

$$M_j^n = \max(u_j^n, u_{j-1}^n) \text{ and } m_j^n = \min(u_j^n, u_{j-1}^n), \tag{5}$$

that is one checks if $m_j^n \leq \overline{u}_j^{n+1} \leq M_j^n$ is true or not. Suppose $\overline{u}_j^{n+1} > M_j^n$, then one has to modify the value of $\overline{u}_j^{n+1}$ and redistribute the mass "around". In the convergence analysis of the method, we discovered that it is better at the theoretical level not to redistribute the mass globally but locally at least in a box of size $p \in \mathbf{N}$ around the current cell. This is why we have introduced a new step in the repair algorithm first proposed in [11] to be able to ensure that the redistribution of mass can be made in the box of size p. Since p is a parameter of the method, one recovers the global repair by setting $p \approx +\infty$.

So let us define boxes of size p. Each box is the collection of cells j such that $rp \leq j \leq (r+1)p - 1$ where $r \in \mathbf{Z}$. The mathematical definition of these boxes B_r is

$$B_r = \{j;\ rp \leq j \leq (r+1)p - 1\}, \quad r \in \mathbf{Z}. \tag{6}$$

However it is also possible to use boxes of different sizes, provided the size is smaller than the predefined maximal box's size p. It is also possible to use moving boxes, that is the starting point of each box is different from one time step to the other. For the simplicity of the mathematical exposure we use only (6).

To make the correction we first need to compute

$$b_r^M = \left(\sum_{j\in B_r} \left(u_j^n - \nu(u_j^n - u_{j-1}^n) - M_j^n\right)\right) - \nu(c_{(r+1)p-\frac{1}{2}}^n - c_{rp-\frac{1}{2}}^n), \tag{7}$$

$$b_r^m = \left(\sum_{j\in B_r} \left(u_j^n - \nu(u_j^n - u_{j-1}^n) - m_j^n\right)\right) - \nu(c_{(r+1)p-\frac{1}{2}}^n - c_{rp-\frac{1}{2}}^n). \tag{8}$$

Most presumably $b_r^M \leq 0$ (resp. $b_r^m \geq 0$), since this is the result of a comparison between $\sum_{j\in B_r}\left(u_j^n - \nu(u_j^n - u_{j-1}^n) - M_j^n\right) \leq 0$ (resp. $\sum_{j\in B_r}\left(u_j^n - \nu(u_j^n - u_{j-1}^n) - m_j^n\right) \geq 0$) and $\nu(c^n_{(r+1)p-\frac{1}{2}} - c^n_{rp-\frac{1}{2}})$. Moreover if p is large enough and ν small enough (i.e. the time step is small) then $b_r^M \leq 0$ (resp. $b_r^m \leq 0$) is probably stating true. The correction is here to ensure that $b_r^M \leq 0$ and $b_r^m \geq 0$ are always satisfied. The idea being that if these inequalities are not satisfied then we multiply the value of the fluxes by a small number such that $b_r^M \leq 0$ and $b_r^m \geq 0$ are fulfilled.

So we define

$$d^n_{j-\frac{1}{2}} = \mu^n_{j-\frac{1}{2}} c^n_{j-\frac{1}{2}} \text{ where } \mu^n_{j-\frac{1}{2}} \in [0,1]. \tag{9}$$

The coefficient $\mu^n_{j-\frac{1}{2}}$ has to be computed to give a corrected value of the flux. The constraint $\mu^n_{j-\frac{1}{2}} \in [0,1]$ appears natural from the consistency point of view. We expect that the definition of these $\mu^n_{j-\frac{1}{2}}$ will be the closest as possible to 1, so that the corrected flux $d^n_{j-\frac{1}{2}}$ is almost equal to the flux of the prediction scheme. We need to check

$$\tilde{b}_r^M - \nu(\mu^n_{(r+1)p-\frac{1}{2}} c^n_{(r+1)p-\frac{1}{2}} - \mu^n_{rp-\frac{1}{2}} c^n_{rp-\frac{1}{2}}) \leq 0, \tag{10}$$

$$\tilde{b}_r^m - \nu(\mu^n_{(r+1)p-\frac{1}{2}} c^n_{(r+1)p-\frac{1}{2}} - \mu^n_{rp-\frac{1}{2}} c^n_{rp-\frac{1}{2}}) \geq 0, \tag{11}$$

where by definition $\tilde{b}_r^M = \sum_{j\in B_r}\left(u_j^n - \nu(u_j^n - u_{j-1}^n) - M_j^n\right)$ is not positive and $\tilde{b}_r^m = \sum_{j\in B_r}\left(u_j^n - \nu(u_j^n - u_{j-1}^n) - m_j^n\right)$ is not negative. One feasible strategy can be derived as:

Analysis of (10)

if $c^n_{(r+1)p-\frac{1}{2}} \geq 0$ **and** $c^n_{rp-\frac{1}{2}} \geq 0$ **:** Then (10) is true once $\nu(\mu^n_{rp-\frac{1}{2}} c^n_{rp-\frac{1}{2}}) \leq -\tilde{b}^M_r$. Thus we define $\varphi^{1,-}_r, \varphi^{1,+}_r$ such that

$$\mu^n_{rp-\frac{1}{2}} \leq \varphi^{1,-}_r = \frac{-\tilde{b}^M_r}{c^n_{rp-\frac{1}{2}}} \text{ and } \varphi^{1,+}_r = +\infty. \tag{12}$$

if $c^n_{(r+1)p-\frac{1}{2}} < 0$ **and** $c^n_{rp-\frac{1}{2}} < 0$ **:** Then (10) is true once $\nu(-\mu^n_{(r+1)p-\frac{1}{2}} c^n_{(r+1)p-\frac{1}{2}}) \leq -\tilde{b}^M_r$. Thus we define $\varphi^{2,-}_r, \varphi^{2,+}_r$ such that

$$\varphi^{2,-}_r = +\infty \text{ and } \mu^n_{(r+1)p-\frac{1}{2}} \leq \varphi^{2,+}_r = \frac{-\tilde{b}^M_r}{-c^n_{(r+1)p-\frac{1}{2}}}. \tag{13}$$

if $c^n_{(r+1)p-\frac{1}{2}} \geq 0$ **and** $c^n_{rp-\frac{1}{2}} < 0$ **:** Then (10) is true without condition. Thus $\varphi^{3,-}_r = \varphi^{3,+}_r = +\infty$

if $c^n_{(r+1)p-\frac{1}{2}} < 0$ **and** $c^n_{rp-\frac{1}{2}} \geq 0$ **:** Then it is not possible to simplify the inequality (10). Thus we impose $\varphi^{4,-}_r = \varphi^{4,+}_r$ and

$$\mu^n_{(r+1)p-\frac{1}{2}},\ \mu^n_{rp-\frac{1}{2}} \leq \varphi^{4,-}_r = \frac{-\tilde{b}^M_r}{c^n_{rp-\frac{1}{2}} - c^n_{(r+1)p-\frac{1}{2}}} \tag{14}$$

Analysis of (11)

if $c^n_{(r+1)p-\frac{1}{2}} \geq 0$ **and** $c^n_{rp-\frac{1}{2}} \geq 0$ **:** Then (11) is true once $\nu(\mu^n_{(r+1)p-\frac{1}{2}} c^n_{(r+1)p-\frac{1}{2}}) \leq \tilde{b}^m_r$. Thus we impose that

$$\psi^{1,-}_r = +\infty \text{ and } \mu^n_{(r+1)p-\frac{1}{2}} \leq \psi^{1,+}_r = \frac{\tilde{b}^m_r}{c^n_{(r+1)p-\frac{1}{2}}}. \tag{15}$$

if $c^n_{(r+1)p-\frac{1}{2}} < 0$ **and** $c^n_{rp-\frac{1}{2}} < 0$ **:** Then (11) is true once $\nu(-\mu^n_{rp-\frac{1}{2}} c^n_{rp-\frac{1}{2}}) \leq \tilde{b}^m_r$. Thus we impose that

$$\mu^n_{rp-\frac{1}{2}} \leq \psi^{2,-}_r = \frac{\tilde{b}^m_r}{-c^n_{rp-\frac{1}{2}}} \text{ and } \psi^{2,+}_r = +\infty. \tag{16}$$

if $c^n_{(r+1)p-\frac{1}{2}} \geq 0$ **and** $c^n_{rp-\frac{1}{2}} < 0$ **:** Then it is not possible to simplify the inequality (11). Thus we impose $\psi^{3,-}_r = \psi^{3,+}_r$ that

$$\mu^n_{(r+1)p-\frac{1}{2}},\ \mu^n_{rp-\frac{1}{2}} \leq \psi^{3,-}_r = \frac{-\tilde{b}^M_r}{-c^n_{(r+1)p-\frac{1}{2}} + c^n_{rp-\frac{1}{2}}} \tag{17}$$

if $c^n_{(r+1)p-\frac{1}{2}} < 0$ **and** $c^n_{rp-\frac{1}{2}} \geq 0$ **:** Then (11) is true without condition. Thus $\psi^{4,-}_r = \psi^{4,+}_r = +\infty$

Let us consider each of the cases considered in inequalities (12) to (17). We gather the restrictions it imposes for all $\mu^n_{rp-\frac{1}{2}}$. The mathematical definition of the correction algorithm is the following.

Definition 2.1 Let us define the corrected fluxes at the boundaries of the boxes

$$d^n_{(r+1)p-\frac{1}{2}} = \min\left(1, \min_l \varphi^{l,+}_r, \min_l \psi^{l,+}_r, \min_l \varphi^{l,-}_{r+1}, \min_l \psi^{l,-}_{r+1}\right) \times c^n_{(r+1)p-\frac{1}{2}}. \tag{18}$$

Inside the boxes we do not correct, that is

$$d^n_{j-\frac{1}{2}} = c^n_{j-\frac{1}{2}}, \quad \forall j \neq rp. \tag{19}$$

The next step consists in the computation of the new prediction $\hat{u}^{n+1}_j$ with the corrected flux:

$$\frac{\hat{u}^{n+1}_j - u^n_j}{\Delta t} + a\frac{u^n_j + d^n_{j+\frac{1}{2}} - u^n_{j-1} - d^n_{j-\frac{1}{2}}}{\Delta x} = 0. \tag{20}$$

LEMMA 2.2 One has the inequalities after the correction step

$$\left(\sum_{j\in B_r}\left(u_j^n-\nu(u_j^n-u_{j-1}^n)-M_j^n\right)\right)-\nu\left(d^n_{(r+1)p-\frac{1}{2}}-d^n_{rp-\frac{1}{2}}\right)\leq 0 \tag{21}$$

$$\left(\sum_{j\in B_r}\left(u_j^n-\nu(u_j^n-u_{j-1}^n)-m_j^n\right)\right)-\nu\left(d^n_{(r+1)p-\frac{1}{2}}-d^n_{rp-\frac{1}{2}}\right)\geq 0. \tag{22}$$

The proof is performed by considering all cases (12-17) separately.

Inequalities (21-22) will be crutial in the analysis of the repairing procedure, essentialy inequality (31) in subsection 2.4.

2.3 Repairing

As already mentioned, to repair a value means successively to compare with the local maximum and minimum, to truncate if needed, then to redistribute the excess of mass on all surrounding cells. Using mathematical notations, one gets

$$\left\{\begin{array}{lll} \text{if } \hat{u}_j^{n+1}>M_j^n & \text{then } \mathbf{u}_j^{n+1}=M_j^n & \text{and } \Delta m_j^n=\hat{u}_j^{n+1}-M_j^n>0,\\ \text{if } \hat{u}_j^{n+1}<m_j^n & \text{then } \mathbf{u}_j^{n+1}=m_j^n & \text{and } \Delta m_j^n=\hat{u}_j^{n+1}-m_j^n<0,\\ \text{else} & \text{then } \mathbf{u}_j^{n+1}=\hat{u}_j^{n+1} & \text{and } \Delta m_j^n=0. \end{array}\right. \tag{23}$$

The total mass in box B_r of the new unknown $\mathbf{u}_j^{n+1}$ may as well be different from the correct mass, so one defines the default of mass as:

$$\Delta M_r^n=\sum_{rp\leq j\leq (r+1)p-1}\Delta m_j^n. \tag{24}$$

This default of mass may be positive or negative. So we need to redistribute it on the box to get at least a conservative algorithm. Following [11], we consider

$$\left\{\begin{array}{ll} \text{if } \Delta M_r^n>0 & \text{then } u_j^{n+1}=\mathbf{u}_j^{n+1}+\lambda_r^n(M_j^n-\mathbf{u}_j^{n+1}),\\ \text{if } \Delta M_r^n<0 & \text{then } u_j^{n+1}=\mathbf{u}_j^{n+1}+\lambda_r^n(m_j^n-\mathbf{u}_j^{n+1}),\\ \text{if } \Delta M_r^n=0 & \text{then } u_j^{n+1}=\mathbf{u}_j^{n+1}, \end{array}\right. \tag{25}$$

where the coefficient λ_r^n is set to

$$\left\{\begin{array}{ll} \text{if } \Delta M_r^n>0 & \lambda_r^n=\frac{\Delta M_r^n}{\sum_{rp\leq j\leq (r+1)p-1}(M_j^n-\mathbf{u}_j^{n+1})},\\ \text{if } \Delta M_r^n<0 & \lambda_r^n=\frac{\Delta M_r^n}{\sum_{rp\leq j\leq (r+1)p-1}(m_j^n-\mathbf{u}_j^{n+1})},\\ \text{if } \Delta M_r^n=0 & \lambda_r^n=0. \end{array}\right. \tag{26}$$

The repair algorithm that we analyze in this paper consists of equations (2) to (26).

2.4 Properties

Before proving our main convergence theorem, we state the stability lemma

LEMMA 2.3 Whatever the value of $p \in \mathbf{N}^\star$ is, for all time step n, the repair algorithm is such that the total mass is preserved (conservation)

$$\sum_j u_j^{n+1} = \sum_j u_j^n, \tag{27}$$

the maxima and minima are respected (maximum principle)

$$m_j^n \leq u_j^{n+1} \leq M_j^n, \quad \forall j, \tag{28}$$

and the total variation is diminishing (TVD)

$$\sum_j |u_j^{n+1} - u_{j-1}^{n+1}| \leq \sum_j |u_j^n - u_{j-1}^n|. \tag{29}$$

Remark: One notices that the scheme is TVD because of (29). **Proof** First one has

$$\sum_{j\in B_r} \hat{u}_j^{n+1} = \left(\sum_{j\in B_r} (u_j^n - \nu(u_j^n - u_{j-1}^n)) \right) - \nu \left(d^n_{(r+1)p-\frac{1}{2}} - d^n_{rp-\frac{1}{2}} \right).$$

Since by construction $\mathbf{u}_j^{n+1} = \hat{u}_j^{n+1} - \Delta m_j^n$, then we get the relation

$$\Delta M_r^n + \sum_{j\in B_r} \mathbf{u}_j^{n+1} = \left(\sum_{j\in B_r} (u_j^n - \nu(u_j^n - u_{j-1}^n)) \right) - \nu \left(d^n_{(r+1)p-\frac{1}{2}} - d^n_{rp-\frac{1}{2}} \right). \tag{30}$$

Assume for instance that $\Delta M_r^n > 0$ (the other cases are easily deduced by mimicking this one). Then the next stage of the algorithm consists in the computation of λ_r^n. The key point is to prove that $0 \leq \lambda_r^n \leq 1$ and the property follows. One has

$$\begin{array}{l} \Delta M_r^n - \sum_{j\in B_r} \left(M_j^n - \mathbf{u}_j^{n+1} \right) \\ = \left(\sum_{j\in B_r} (u_j^n - \nu(u_j^n - u_{j-1}^n) - M_j^n) \right) - \nu \left(d^n_{(r+1)p-\frac{1}{2}} - d^n_{rp-\frac{1}{2}} \right), \end{array}$$

and clearly

$$\Delta M_r^n - \sum_{j\in B_r} \left(M_j^n - \mathbf{u}_j^{n+1} \right) \leq 0, \tag{31}$$

due to the correction step (21)-(22). Then $\Delta M_r^n \leq \sum_{j\in B_r} \left(M_j^n - \mathbf{u}_j^{n+1} \right)$ that is

$$0 \leq \lambda_r^n = \frac{\Delta M_r^n}{\sum_{j\in B_r} M_j^n - \mathbf{u}_j^{n+1}} \leq 1.$$

And finally

$$m_j^n \leq u_j^{n+1} = (1 - \lambda_r^n)\mathbf{u}_j^{n+1} + \lambda_r^n M_j^n \leq M_j^n,$$

since $m_j^n \leq \mathbf{u}_j^{n+1} \leq M_j^n$ by construction. Thus (28) is proved for $\Delta M_r^n > 0$. (The other case is proved using the same method.) It implies (29).
It remains to prove (27). The definition of λ_r^n associated with equation (30) yields to

$$\begin{aligned}\sum_{j\in B_r} u_j^{n+1} &= \sum_{j\in B_r} \mathbf{u}_j^{n+1} + \lambda_r^n \sum_{j\in B_r} \left(M_j^n - \mathbf{u}_j^{n+1}\right) \\ &= \sum_{j\in B_r} \mathbf{u}_j^{n+1} + \Delta M_r^n = \sum_{j\in B_r} (u_j^n - \nu(u_j^n - u_{j-1}^n)) - \nu(d^n_{(r+1)p-\frac{1}{2}} - d^n_{rp-\frac{1}{2}}).\end{aligned}$$

Summing up with respect to r gives (27) and completes the proof. ■

2.5 Finite Volume Form

The next step is to identify the algorithm (2) to (26) as a standard finite volume scheme as stated in (35). For the simplicity of notations, assume that $\Delta M_r^n > 0$. Due to the definition of the scheme (2) to (26), one has

$$\begin{aligned}u_j^{n+1} &= \mathbf{u}_j^{n+1} + \lambda_r^n(M_j^n - \mathbf{u}_j^{n+1}) = \hat{u}_j^n - \Delta m_j^n + \lambda_r^n(M_j^n - \mathbf{u}_j^{n+1}) \\ &= \left(u_j^n - \nu(u_j^n - u_{j-1}^n)\right) - \nu(d^n_{j+\frac{1}{2}} - d^n_{j-\frac{1}{2}}) + \left(-\Delta m_j^n + \lambda_r^n(M_j^n - \mathbf{u}_j^{n+1})\right).\end{aligned}$$

In order to be able to rewrite the scheme as a finite volume scheme, we need to construct some fluxes $e^n_{j+\frac{1}{2}}$ such that for all j

$$\Delta t(e^n_{j+\frac{1}{2}} - e^n_{j-\frac{1}{2}}) = -\Delta m_j^n + \lambda_r^n(M_j^n - \mathbf{u}_j^{n+1}). \tag{32}$$

The solution can be constructed as

$$\begin{cases} e^n_{rp-\frac{1}{2}} = 0, & \forall r \in \mathbf{Z}, \\ e^n_{j+\frac{1}{2}} = \frac{1}{\Delta t}\sum_{rp\leq k\leq j}\left(-\Delta m_k^n + \lambda_r^n(M_k^n - \mathbf{u}_k^{n+1})\right), & rp < j. \end{cases} \tag{33}$$

This formula is correct because (24)-(26) implies that (33) is correct for $j = (r+1)p - 1$: $e^n_{(r+1)p-\frac{1}{2}} = 0$. If one (or more) of the $\Delta M_r^n < 0$ is negative, the result remains the same.

Lemma 2.4 The L^1 norm of $(e^n_{j+\frac{1}{2}})$ is bounded

$$||e^n||_1 = \Delta x \sum_j |e^n_{j+\frac{1}{2}}| \leq 2C_1(2k+1)p||u||_{BV}, \tag{34}$$

where $||u||_{BV}$ is the BV norm of the solution.

Proof Due to the definition (33) we deduce

$$||e^n||_1 \leq \frac{p}{\Delta t}\Delta x \sum_j |\Delta m_j^n|.$$

But one always has

$$\begin{aligned} |\Delta m_j^n| \le |\hat{u}_j^{n+1} - (u_j - \nu(u_j^n - u_{j-1}^n))| &\le \nu(|d_{j+\frac{1}{2}}^n| + |d_{j-\frac{1}{2}}^n|) \\ &\le \nu(|c_{j+\frac{1}{2}}^n| + |c_{j-\frac{1}{2}}^n|), \end{aligned}$$

moreover due to the hypothesis (4) we finally get

$$\frac{\Delta x}{\Delta t}\sum_j |\Delta m_j^n| \le 2C_1(2k+1)\sum_j |u_j^n - u_{j-1}^n| \le 2C_1(2k+1)||u||_{BV}.$$

This ends the proof. ■

Then we are able to write the repair algorithm as

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_j^{n-1}}{\Delta t} = (s_j^n - s_{j-1}^n), \tag{35}$$

where s_j^n is defined by

$$s_j^n = -\frac{1}{\Delta x} a d_{j+\frac{1}{2}}^n + e_{j+\frac{1}{2}}^n. \tag{36}$$

The L^1 norm of s^n is bounded by

$$||s^n||_1 \le C_1(2k+1)||u||_{BV} + ||e^n||_1 \le C_1(2k+1)(2p+1)||u||_{BV}. \tag{37}$$

From (35) we get

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{a}{\Delta x}(I - T)u^n = (I - T)s^n, \tag{38}$$

where T is the translation operator to the right. This is equivalent to say that

$$u^n - (I + \nu(T - I))^n u^0 = \Delta t \sum_{p=0}^{n-1} (I + \nu(T - I))^{n-p-1}(I - T)s^p. \tag{39}$$

The main convergence result is given by the

THEOREM 2.5 Assume that $\nu < 1$. The difference between the upwind scheme and the repair algorithm (as defined by (2) to (26), or (35-36)) tends to zero with Δx in the sense that there exists a constant $C_2 > 0$ which does not depend on k, p, C_1, such that

$$||u^n - (I + \nu(T - I))^n u^0||_1 \le C_2 C_1(2k+1)(2p+1)||u||_{BV}\sqrt{(n\Delta t)\Delta x}. \tag{40}$$

Proof Equation (38-39) shows that the scheme exactly fits in the framework developed in [7] and applied in [8] for the convergence analysis of non linear schemes for linear advection. The key estimate proved in [7] is that

$$||(I + \nu(T - I))^{q+1}(I - T)||_1 \le C\frac{1}{\nu(1-\nu)}\frac{1}{\sqrt{q}}. \tag{41}$$

The constant C is a universal one. A simplified proof is provided in appendix for the case $\nu = \frac{1}{2}$. This estimate means that the right hand side of (39) is made of terms that are small with respect to $n - p$. The final stage of the proof is straightforward. One has

$$||u^n - (I + \nu(T - I))^n u^0||_1 \leq \Delta t \max ||s^p||_1 \times \sum_{p=0}^{n-1} ||(I + \nu(T - I))^{n-p-1}(I - T)||_1$$

$$\leq C \frac{1}{\nu(1-\nu)} \Delta t \max ||s^p||_1 \sum_{q=1}^{n} \frac{1}{\sqrt{q}} \leq C \frac{1}{\nu(1-\nu)} \Delta t \max ||s^p||_1 \int_0^n \frac{dy}{\sqrt{y}}$$

$$\leq C \frac{1}{\nu(1-\nu)} \Delta t C_1 (2k+1)(2p+1)||u||_{BV} \frac{\sqrt{n}}{2}$$

$$\leq \frac{C}{2a\sqrt{1-\nu}} C_1 (2k+1)(2p+1)||u||_{BV} \sqrt{T \Delta x}.$$

Defining $C_2 = \frac{C}{2a\sqrt{1-\nu}}$ ends the proof. ■

Remark: Since the numerical solution of the upwind scheme converges to the exact solution meaning

$$(I + \nu(T - I))^n u^0 \to u(n\Delta t) \quad \text{in } L^1(\mathbf{R}) \text{ as } \Delta x \to 0,$$

then the inequality of theorem 2.5 is a convergence result. Of course this inequality does not explain that the repair algorithm is better than the upwind scheme. Actually Repair is equivalent to the upwind scheme is one choose $c^n_{j+\frac{1}{2}} = 0$, for all j and n. However the result shows that repair can not diverge if $p^2 \Delta x \to 0$.

Remark: On the other hand the error estimate blows up if p is too large. At least we need $p^2 \Delta x \to 0$ to get a vanishing error on the right hand side of the estimate. This estimate is the reason why we have incorporated the correction step in the repair algorithm.

Remark: One may wonder the reason of the $\frac{1}{a\sqrt{1-\nu}}$ contribution in the definition of the constant C_2. Indeed if $a \to 0$ or $\nu \to 1$, then C_2 can go to infinity which makes the estimate of convergence meaningless. First of all, for a given computation $C_2 < \infty$. Second of all, it is possible to use sharper estimates to get rid of the $\frac{1}{a}$. This is done in [7] for the convergence analysis of TVD schemes for instance. Finally one may argue that a similar argument should be possible at the theoretical level to get rid of the $\frac{1}{\sqrt{1-\nu}}$. Nevertheless real computations use $\nu < 1$ which is another reason to use this hypothesis. At the numerical level, we never saw any dependence of the rate of convergence with respect to this parameter. Thus we consider this as an artifact of the analysis.

3 Repair algorithm for remapping

In this section we show how to extend the previous result to take into account some very simple remapping algorithms. The main idea is that remapping is very close

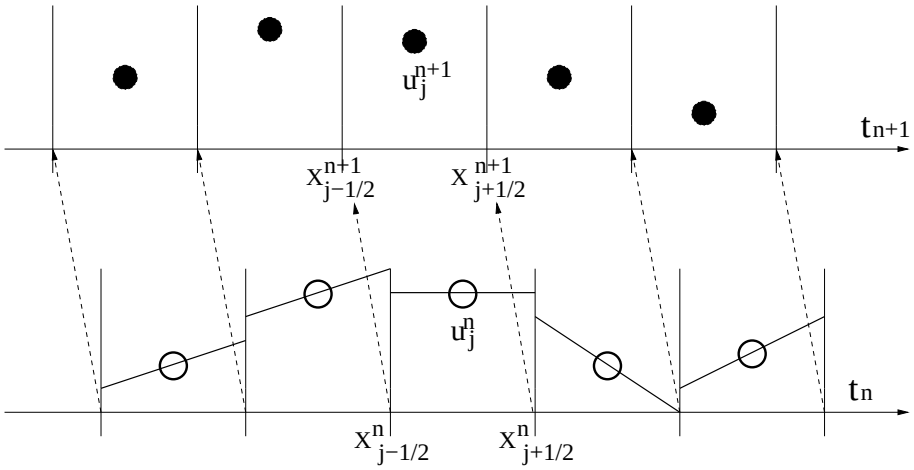

Figure 1: Remapping stage. The old and new densities are the bullets. The *high order* reconstruction stage are represented with the slopes.

to transport. So let us consider a uniform mesh. (The uniformity of the mesh is not absolutely necessary however this hypothesis simplifies a lot the proof.).

At time step n, the value of the unknown is u_j^n in the cell j. The boundaries of cell j are $x^n_{j-\frac{1}{2}}$ and $x^n_{j+\frac{1}{2}} = x^n_{j-\frac{1}{2}} + \Delta x$, where Δx is the mesh size. Then the mesh changes. We assume the simplest change: a uniform translation to the left, then

$$x^{n+1}_{j+\frac{1}{2}} = x^n_{j+\frac{1}{2}} - a\Delta t, \quad \forall j, \tag{42}$$

where $a > 0$ is a kind of mesh velocity and Δt is an equivalent time step. We assume that the mesh does not move too fast, that is we assume the CFL condition

$$\nu = a\frac{\Delta t}{\Delta x} \leq 1.$$

A standard remapping algorithm is divided in two stages : first reconstruct a *high order* profile using a MUSCL like algorithm, then project on the new mesh. Let us detail these operations. An illustration is given in figure 1.

The reconstruction amounts to the definition of the slopes in function of the old values. So we define

$$v_j^n(x) = u_j^n + w_j^n(x), \tag{43}$$

where w_j^n is the slope function which is added to the average value at step n. A natural hypothesis is that the average value of $w_j^n(x)$ is zero

$$\int_{x^n_{j-\frac{1}{2}}}^{x^n_{j+\frac{1}{2}}} w_j^n(x) = 0, \tag{44}$$

and that $w_j^n(x)$ is a local reconstruction, that is

$$|w_j^n(x)| \leq C_3 \sum_{j-k\leq q\leq j+k} |u_q^n - u_{q-1}^n|. \tag{45}$$

Then we project the reconstructed function onto the new mesh. Thus

$$u_j^{n+1} = \int_{x_{j-\frac{1}{2}}^{n+1}}^{x_{j+\frac{1}{2}}^{n+1}} v^n(x)dx. \tag{46}$$

That is

$$u_j^{n+1} = \int_{x_{j-\frac{1}{2}}^{n+1}}^{x_{j-\frac{1}{2}}^{n}} v_{j-1}^n(x)dx + \int_{x_{j-\frac{1}{2}}^{n}}^{x_{j+\frac{1}{2}}^{n+1}} v_j^n(x)dx.$$

Replacing $v_j^n(x)$ and $v_{j-1}^n(x)$ using (43), one gets

$$u_j^{n+1} = \nu u_{j-1}^n + (1-\nu)u_j^n + \int_{x_{j-\frac{1}{2}}^{n+1}}^{x_{j-\frac{1}{2}}^{n}} w_{j-1}^n(x)dx + \int_{x_{j-\frac{1}{2}}^{n}}^{x_{j+\frac{1}{2}}^{n+1}} w_j^n(x)dx.$$

Let us define

$$c_{j-\frac{1}{2}}^n = -\frac{1}{x_{j-\frac{1}{2}}^n - x_{j-\frac{1}{2}}^{n+1}} \int_{x_{j-\frac{1}{2}}^{n+1}}^{x_{j-\frac{1}{2}}^{n}} w_{j-1}^n(x)dx = -\frac{1}{a\Delta t} \int_{x_{j-\frac{1}{2}}^{n+1}}^{x_{j-\frac{1}{2}}^{n+1}+a\Delta t} w_{j-1}^n(x)dx. \tag{47}$$

Hypothesis (45) turns into

$$|c_{j+\frac{1}{2}}^n| \leq C_3 \sum_{j-k\leq q\leq j+k} |u_q^n - u_{q-1}^n|. \tag{48}$$

Moreover using (44) — the fact that the mean value of the correction is zero — yields to the definition of the new value $\overline{u}_j^{n+1}$

$$\frac{\overline{u}_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n + c_{j+\frac{1}{2}}^n - u_{j-1}^n - c_{j-\frac{1}{2}}^n}{\Delta x} = 0. \tag{49}$$

With these notations the remapping algorithm is equivalent to a non linear transport. Thus we easily generalize the results of the previous section to remapping.

The repair algorithm for remapping is then :

1) reconstruct a *high order* approximation as in (43-45) ;
2) make the correction step as in subsection 2.2 ;
3) repair.

The generalization of the correction step of subsection 2.2 is easy to write down, since the notations are quite similar between section 2 and section 3. It is sufficient to notice that once $p \in \mathbf{N}^\star$, the size of the boxes, has been chosen, then the correction step checks if the inequalities (7-8) become true. If one of them is not true, then we need to minimize the value of $c_{j+\frac{1}{2}}^n$ at the borders of the boxes until the inequalities are true.

Then the convergence result of theorem 2.5 is extended to Repair remapping algorithms. For a given size of the boxes p, the difference between the Repair Algorithm and the upwind or donor cell method is bounded by estimate (40) (with $C_1 = C_3$).

4 Numerical results

In the numerical result presented, we tried the Lax-Wendroff scheme plus Repair and the Downwind scheme plus Repair. The velocity is $a = 1$.

The Lax-Wendroff scheme is given by

$$\textbf{(LW)} \quad \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_j^n + \frac{1}{2}(1-\nu)(u_{j+1}^n - u_j^n) - u_{j-1}^n + \frac{1}{2}(1-\nu)(u_j^n - u_{j-1}^n)}{\Delta x} = 0.$$

The Downwind scheme is

$$\textbf{(DW)} \quad \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+1}^n - u_j^n}{\Delta x} = 0.$$

The LW intends to be representative of high-order prediction schemes. The DW scheme intends to be representative of highly anti-dissipative prediction schemes. Concerning the Repair algorithm we used different box sizes.
We advected two initial profiles: a smooth one ($u_0(x) = -\cos 2\pi x$ on $[0, 1]$) and a discontinuous one ($u_0(x) = 1$ for $0 < x < 0.5$ and $u_0(x) = 0$ for $0.5 < x < 1$) on a domain with periodic boundary conditions.

4.1 Figures

Essentially the results LW+Repair show good convergence properties independently to the size of the box p. See Figures 2 and 3. However one sees for the cosine initial condition that $p = 1$ leads to some discrepancy at the extrema of the solution, that $p = 50, 100$ leads to a prediction of the extrema that is less accurate. On the other hand a relative small box of size $p = 4$ gives the best result. On this case, a small amount of local repairing gives good results. For the step initial condition of Figure 3, the spreading is quite the same with all box sizes.

On the other hand the DW+repair results may be very sensitive to the size of the box. For a given number of cells, one gets the best results for the smaller size of the box, see Figure 4, staircases appear, exactly as with the UltraBee scheme as reported in [6], see also [2] and [12]. For a given box size the numerical solution converges to the exact one as the number of cells increases, see Figure 5. Unfortunately the global algorithm where the size of the box is equal to the number of cells (i.e. $p = n$) seems to diverge as the time increases, see Figure 6. Our experiments showed that this pathology is highly sensitive to the CFL number. For $CFL = .5$, the global algorithm seems to be correct. We retain that the global algorithm is not reliable for such test cases.

4.2 Order of convergence

In tables 1 to 3 are gathered some errors in various norms. The size of the boxes is $p = 4$. The final time is $t = 1$. We increase the number of cells from 100 to 1600. The order of convergence of the Lax-Wendroff plus Repair algorithm is $\approx \frac{1}{2}$ in L^1 for a discontinuous profile. This is in accordance with the theoretical result of Theorem 2.5. For a smooth profile the order is ≈ 2 in all norms.

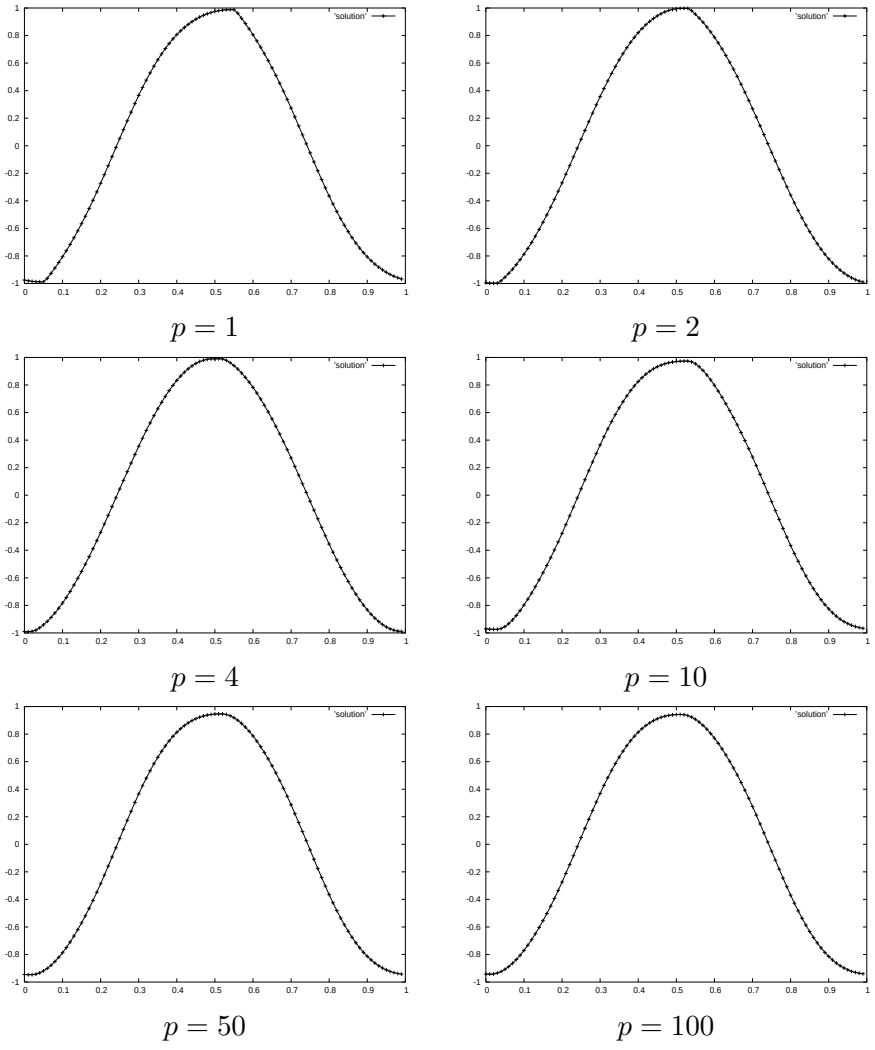

Figure 2: Lax-Wendroff plus Repair : 100 cells. The initial condition is $u_0(x) = -\cos 2\pi x$. The size of the box is $p = 1, 2, 4, 10, 50, 100$ from top left to bottom right. The final time is $t = 10$

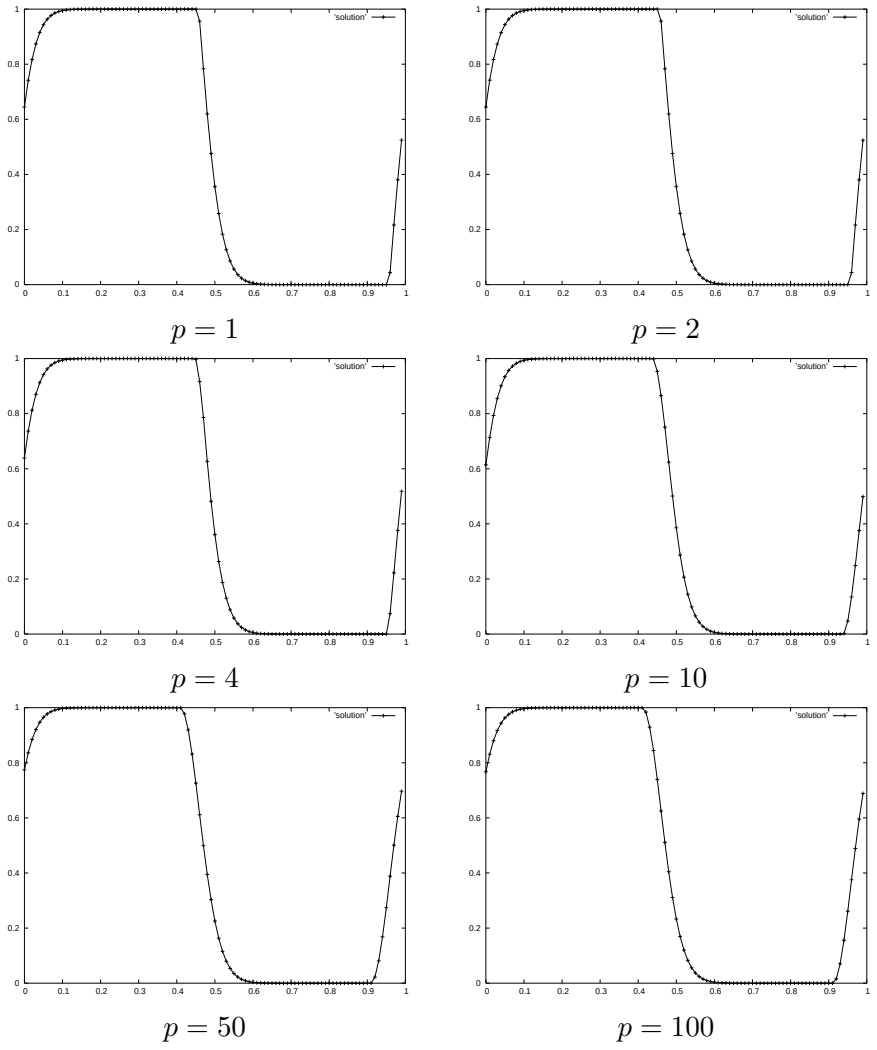

Figure 3: Lax-Wendroff plus Repair : 100 cells. The initial condition is $u_0(x) = 1$ for $0 < x < 0.5$ and $u_0(x) = 0$ for $0.5 < x < 1$. The size of the box is $p = 1, 2, 4, 10, 50, 100$ from top left to bottom right. The final time is $t = 1$

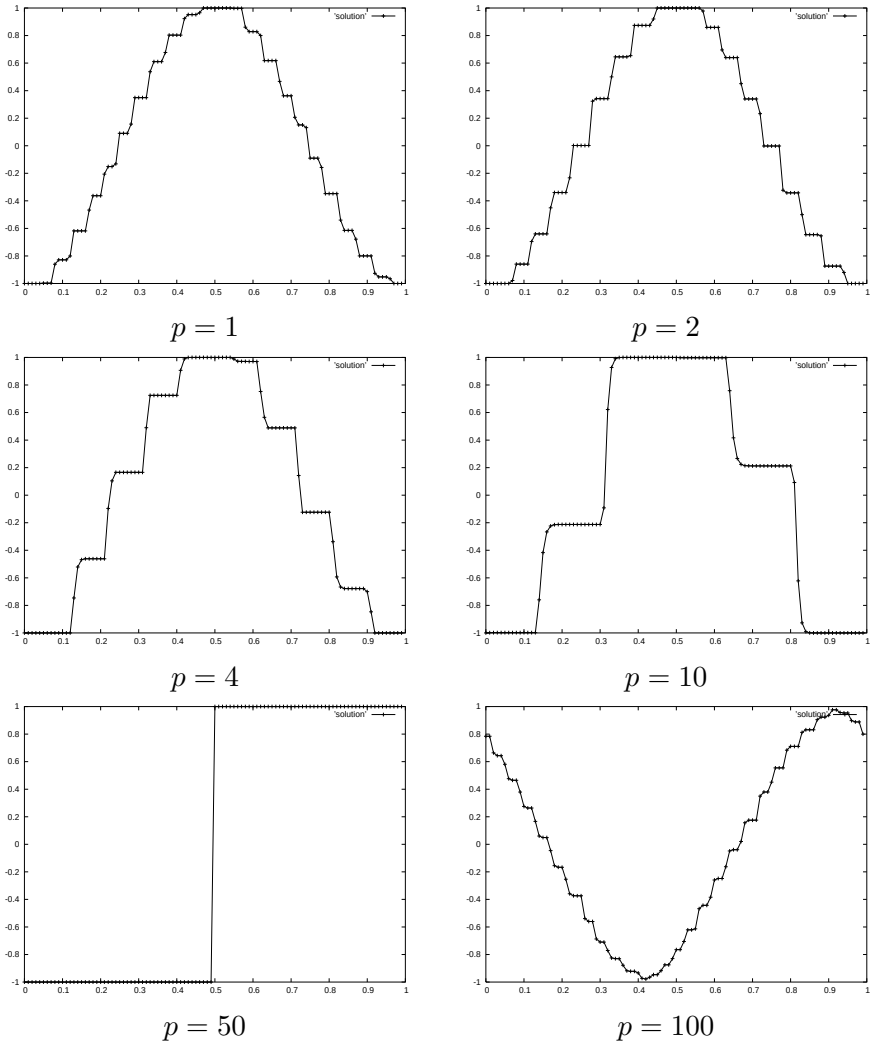

Figure 4: Downwind plus Repair. The initial condition is $u_0(x) = -\cos 2\pi x$. The size of the box is $p = 1, 2, 4, 10, 50, 100$ from top left to bottom right. The number of cells is $N = 100$. The Courant number is $CFL = 0.12345$. The final time is $t = 1$

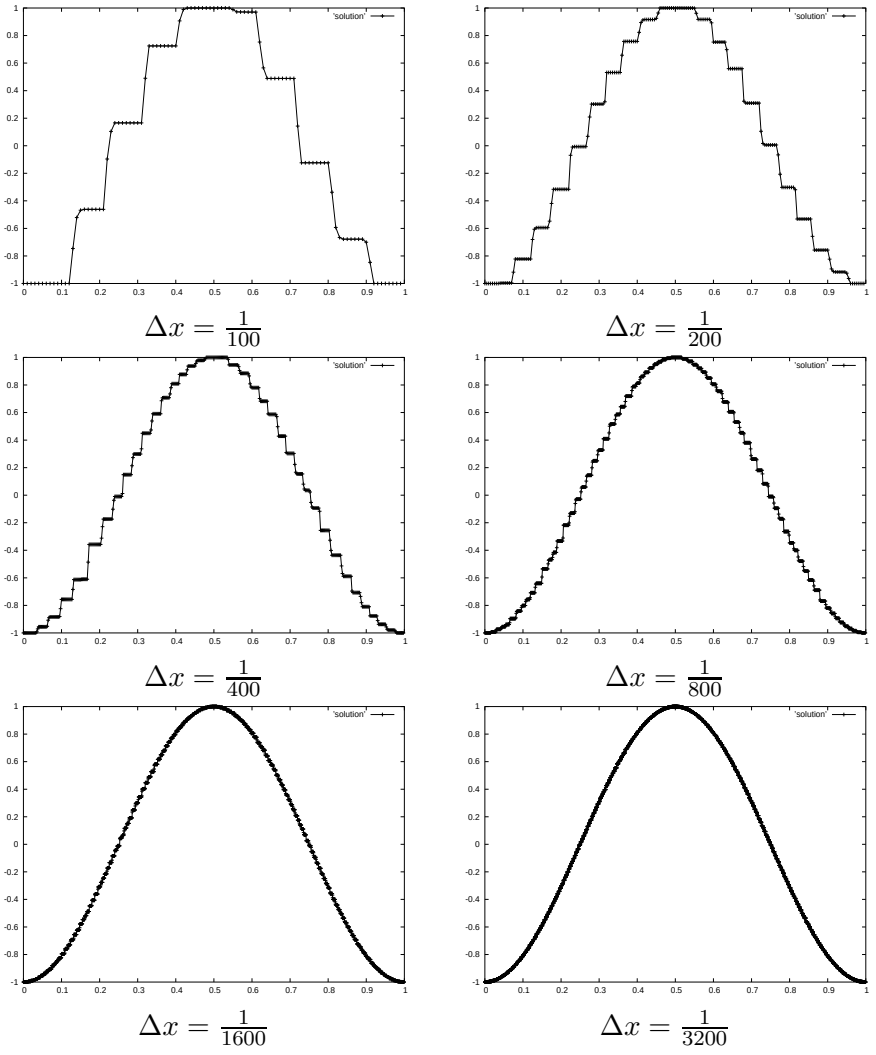

Figure 5: Downwind plus Repair. The initial condition is $u_0(x) = -\cos 2\pi x$. The size of the box is $p = 4$. The number of cells is $n = 100, 200, 400, 800, 1600, 3200$. The final time is $t = 1$

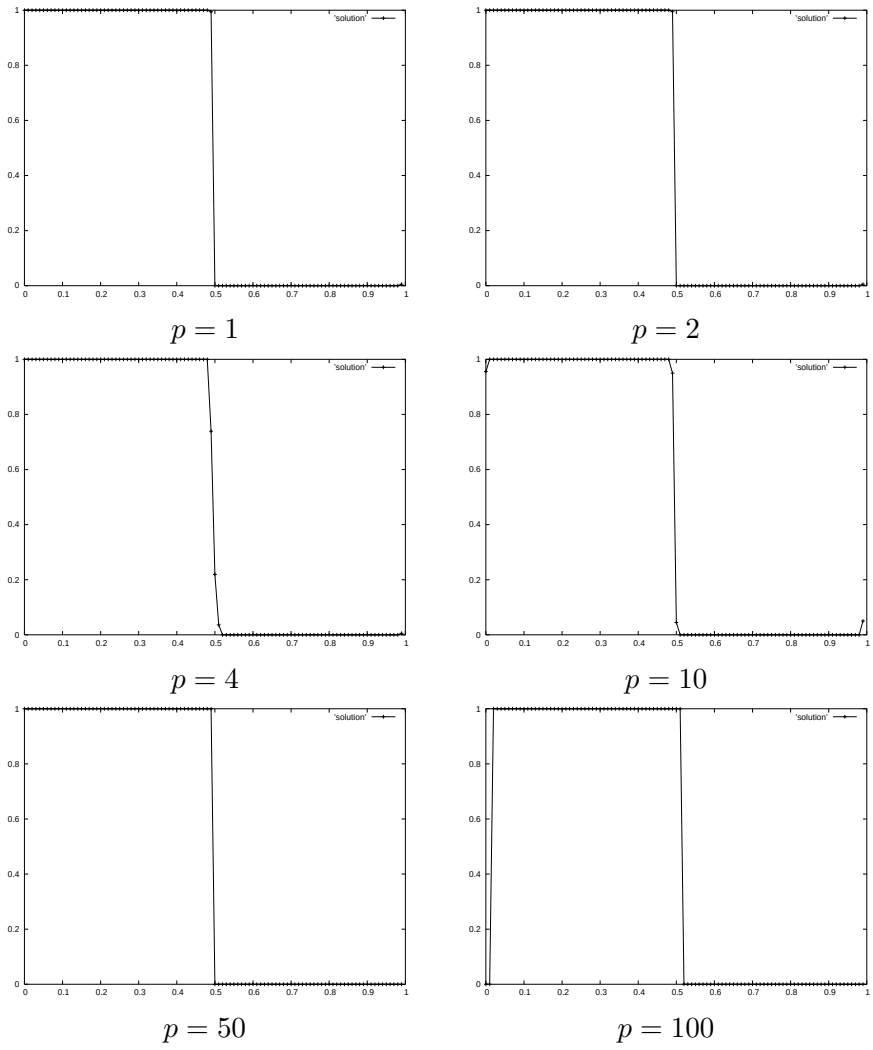

Figure 6: Downwind plus Repair : 100 cells. The initial condition is $u_0(x) = 1$ for $0 < x < 0.5$ and $u_0(x) = 0$ for $0.5 < x < 1$. The size of the box is $p = 1, 2, 4, 10, 50, 100$ from top left to bottom right. The final time is $t = 1$. The results are quite good, except for the global algorithm which seems to be in advance.

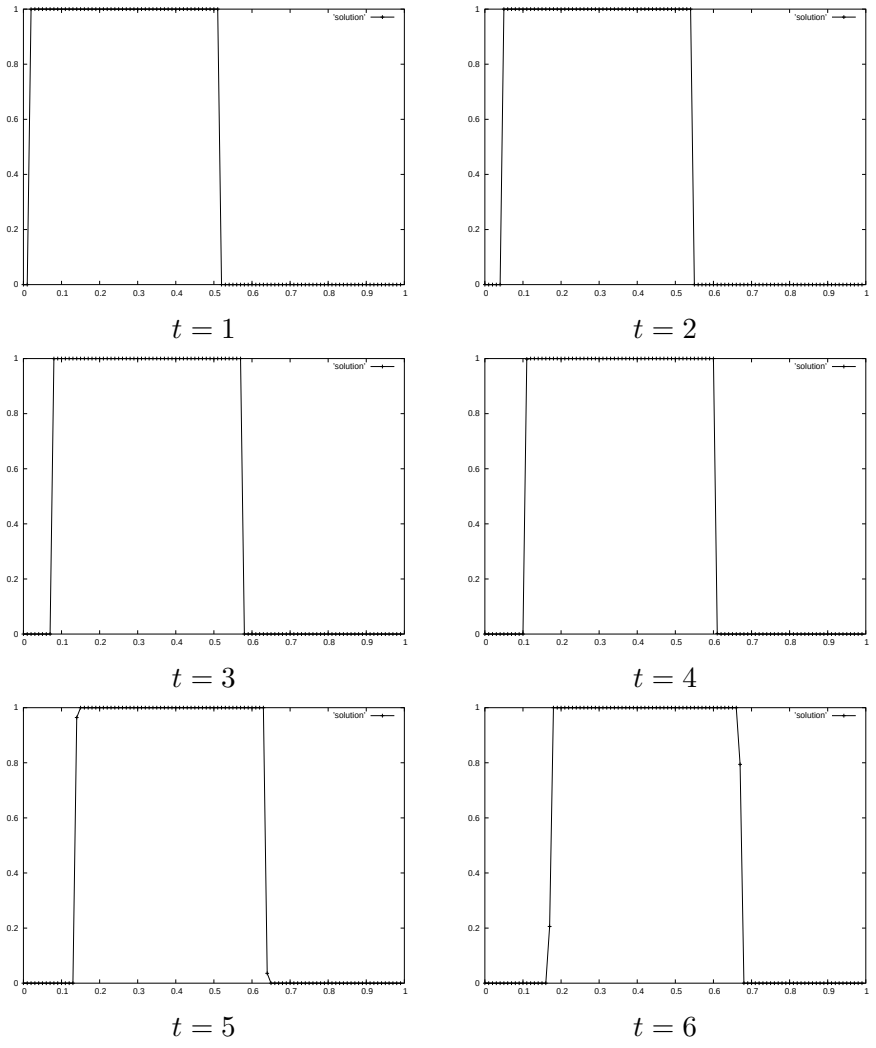

Figure 7: Downwind plus Repair : 100 cells. The initial condition is $u_0(x) = 1$ for $0 < x < 0.5$ and $u_0(x) = 0$ for $0.5 < x < 1$. The size of the box is $p = 100$. The final time is $t = 1, 2, 3, 4, 5, 6$. The global algorithm is in advance.

cells	L^1	L^2	L^∞
100	0.00285	0.00306	0.00469
200	0.000705	0.000762	0.00149
400	0.000172	0.000187	0.000416
800	0.0000425	0.0000426	0.000123
1600	0.0000105	0.0000115	0.0000322
order	≈ 2	≈ 2	≈ 2

Table 1: Errors with respect to the size of the mesh. Lax-Wendroff plus Repair $p = 4$. The initial condition is $u_0(x) = -\cos 2\pi x$. The order is approximatively 2. The Courant number is $CFL = 0.25$. Compare with figure 2.

cells	L^1	L^2	L^∞
100	0.0475	0.120	0.518
200	0.0297	0.0959	0.534
400	0.0187	0.0762	0.547
800	0.0117	0.0607	0.557
1600	0.00741	0.0483	0.566
order	$\approx \frac{1}{2}$	$\approx \frac{1}{4}$	≈ 0

Table 2: Errors with respect to the size of the mesh. Lax-Wendroff plus Repair $p = 4$. The initial condition is $u_0(x) = 1$ for $0 < x < 0.5$ and $u_0(x) = 0$ for $0.5 < x < 1$. The final time is $t = 1$. The Courant number is $CFL = 0.12345$. Compare with figure 3.

The order of convergence of the Downwind plus Repair algorithm is approximatively one in all norms for a smooth profile. It seems that Downwind plus Repair *finds* the closest step profile to the initial profile in a few time steps. Therefore Downwind plus Repair is almost exact for a step profile. This is very similar with what has been reported in [6] for the Downwind TVD scheme (also known in the literature as the Ultra-Bee scheme).

For the discontinuous profile $u_0(x) = 1$ for $0 < x < 0.5$ and $u_0(x) = 0$ for $0.5 < x < 1$, the result are almost exact for 200 and more cells.

5 Gas dynamics in 1D

In this section we are solving the gas dynamics equations in 1D with a Lagrange+Remap code. This code is built on two components: a Lagrangian scheme and a Remap strategy which may need repair.
The primitive variables are ρ the density, u the velocity and ε the specific internal energy, whereas, the conservative variables are m the mass, $\mu = mu$ the momentum and $E = \mathcal{E} + K = m\varepsilon + \frac{1}{2}mu^2$ the total energy (as the internal energy plus the kinetic energy). Density and specific internal energy are given on cell centers whereas velocity is given on nodes.

cells	L^1	L^2	L^∞
100	0.090947985213994	0.115785613616216	0.2710309
200	0.041511107024698	0.051893396383501	0.124615354
400	0.023674021241459	0.031497599561389	0.101446636
800	0.012277712311722	0.016193499317599	0.052800138
1600	0.006085189480290	0.008036148646353	0.029193723
order	≈ 1	≈ 1	≈ 1

Table 3: Errors with respect to the size of the mesh. Downwind plus Repair. The initial condition is $u_0(x) = -\cos 2\pi x$. The order is approximatively 1. Compare with figure 5.

The Lagrangian scheme is based on a compatible discretisation as described in [5], [4] and [3]. This Lagrangian scheme is associated with a high-order remapping algorithm based on [9]. This remapping algorithm is conservative in mass, momentum and total energy, reversible (if the old and new meshes are identical, the primitive variables remain unchanged) and high-order as a piecewise linear reconstruction is performed for each variable before remapping as [9].

When solving the Euler equations we have to ensure that the density, the specific internal energy and the pressure remain positive during the calculation. Our Lagrangian scheme ensures this property whereas the remapping algorithm does not necessarily: a Repair method has then been added. Moreover the control of spurious oscillations is performed because a maximum principle is enforced by the repair method which was not an intrinsic property of the remapping method (see [10]).
The repair process has to be embedded into the remapping process as the primitive variables (ρ, u, ε) are not independent from the conservative variables (m, μ, E); for example momentum is computed using velocity (which may need repair) and mass. However the mass is computed using the density which may need repair as well. Therefore the Remap+Repair algorithm performs the following three steps:

1. Remap mass, then repair density — mass and density are final values;
2. Remap momentum, then repair velocity — momentum and velocity are final values;
3. Remap internal energy and kinetic energy, then repair specific internal energy — energies are final values.

The repair algorithm finally brings several new properties to the overall scheme, the respect of positivity and the preservation of a maximum principle.
It is well known that any numerical scheme which does not preserve the positivity by construction has to have a "kind" of non-conservative repair method such as $\tilde{\rho} = \max(0, \rho)$ and $\tilde{\varepsilon} = \max(0, \varepsilon)$ to exactly ensure positivity of density and specific internal energy. However we believe that a repair has to be performed more carefully especially to ensure the conservation.
The problems chosen to test the method in 1D are the Sod Riemann problem, the

cells	L^1	L^2	L^∞
100	0.007557530	0.01467060	0.0859691
200	0.004063840	0.00905263	0.0589657
400	0.002238250	0.00622884	0.0570818
800	0.001238270	0.00452951	0.0678852
order	≤ 1	≤ 0.5	0

Table 4: Errors for density with respect to the size of the mesh. Sod Riemann problem — Euler regime (as Lagrange+Remap) with Repair process.

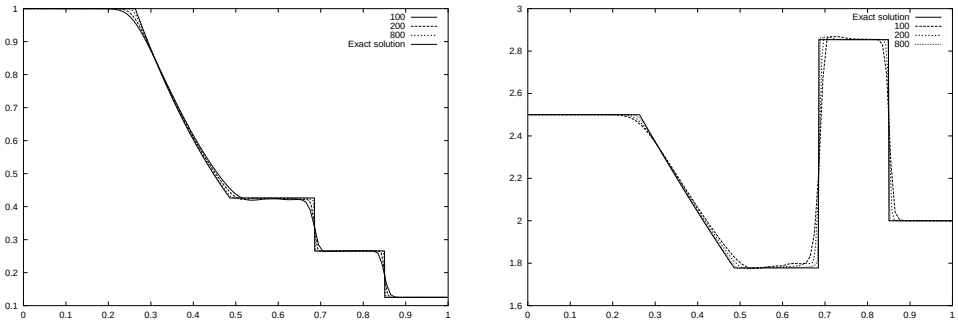

Figure 8: Sod Riemann problem — Euler regime (as Lagrange+Remap) with Repair process — Density for a $100, 200, 800$ mesh sizes *vs* exact solution.

Le Blanc Riemann problem and the Collela-Woodward blastwave. For each of these problems the code is run in Euler regime as Lagrange+Remap with a local repair method.

5.1 Sod Riemann problem

The Sod Riemann problem is defined on $\Omega = [0; 1]$ by two states of the same ideal gas $\gamma = 1.4$, separated at $X = \frac{1}{2}$: $(\rho_l, u_l, p_l) = (1, 0, 1)$ and $(\rho_r, u_r, p_r) = (0.125, 0, 0.1)$. The CFL is chosen equal to 0.25. In table 4 are gathered the convergence results in L^p norms ($p = 1, 2, \infty$) for mesh sizes from 100 to 1600.

In Figure 8 are presented the density and the specific internal energy *versus* the exact solution for several mesh sizes $100, 200, 800$. The Sod Riemann problem is a mild problem in the sense that the repair process can be canceled without generating any problem on the numerical solution. On the other hand this problem shows that a repair method, even not needed, is not polluting the numerical solution.

5.2 Le Blanc Riemann problem

The Le Blanc Riemann problem is defined on $\Omega = [0; 9]$ by two states of the same fluid $\gamma = \frac{5}{3}$, separated at $X = 3$: $(\rho_l, u_l, p_l) = (1, 0, \frac{2}{3}10^{-1})$ and $(\rho_r, u_r, p_r) = (10^{-3}, 0, \frac{2}{3}10^{-9})$. The CFL is chosen equal to 0.25. In table 5 is gathered the convergence results in L^p norms ($p = 1, 2, \infty$) for a mesh size from 180 to 2880. In

cells	L^1	L^2	L^∞
180	0.0698808	0.0894762	0.198472
360	0.0148312	0.0107787	0.0330398
720	0.00909984	0.00849729	0.040686
1440	0.00507182	0.00586531	0.0406224
2880	0.00278987	0.00431683	0.0426995
order	≤ 1	≤ 0.5	0

Table 5: Errors for specific internal energy with respect to the size of the mesh — Le Blanc Riemann problem — Euler regime (as Lagrange+Remap) with Repair process.

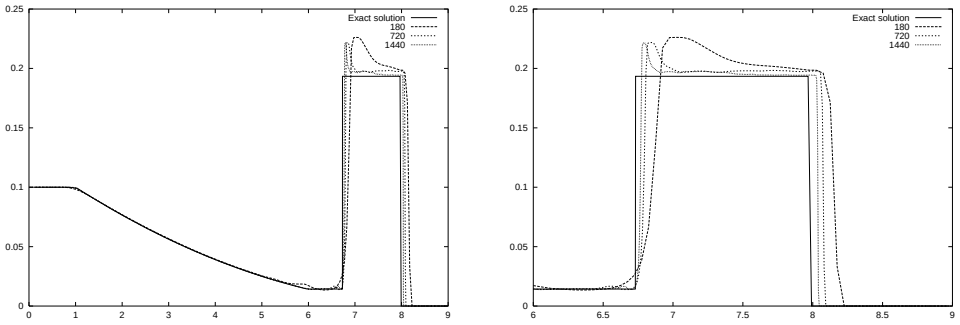

Figure 9: Le Blanc Riemann problem — Euler regime (as Lagrange+Remap) with Repair process — Specific internal energy for a $180, 720, 1440$ mesh sizes *vs* exact solution — Full size (left) and zoom (right).

this problem the specific internal energy simulation is the most difficult part because most of the numerical schemes (Eulerian or Lagrangian) converge very slowly on this problem. In Figure 9 are presented the energy for different mesh sizes (180,720 and 1440). The Le Blanc Riemann problem is a more severe test case where the repair method is necessary, otherwize negative internal energies are created and the code fails if no special treatment is provided. Moreover if only positivity is ensured then the overshoot observed in Figure 9 is much more pronounced.

5.3 Collela-Woodward blastwave

The Collela-Woodward blastwave problem simulates on $\Omega = [0, 1]$ the interaction of two Riemann problems defined by the three states $(\rho_l, u_l, p_l) = (1, 0, 1000)$, $(\rho_m, u_m, p_m) = (1, 0, 0.001)$, $(\rho_r, u_r, p_r) = (1, 0, 100)$ of the same ideal fluid $\gamma = 1.4$ separated at $X_1 = 0.1$ and $X_2 = 0.9$. The CFL is chosen to be equal to 0.7. In figure 10 are presented the density for $200, 400, 1600$ mesh sizes compared to a reference solution showing numerically the convergence of the method. This problem, as the Le blanc Riemann problem, can not run without a repair method. The repair process actually fixes the unphysical densities and specific internal energies and ensures

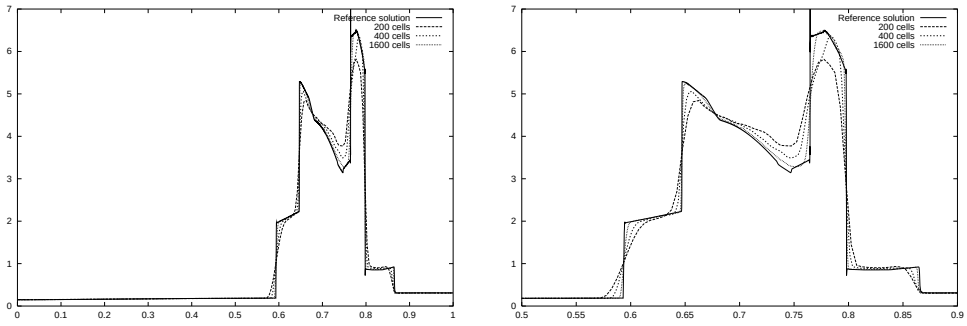

Figure 10: Collela-Woordward blastwave problem — Euler regime (as Lagrange+Remap) with Repair process — Density for $200, 400, 1600$ mesh sizes *vs* reference solution — Full size (left) and zoom (right).

a maximum principle[1].

6 Conclusion

In this paper we investigated the convergence in 1D of a repair algorithm. A Repair algorithm can be defined as a post-processing to conservatively ensure a maximum principle (hence the positivity) for any numerical scheme. In order to prove the main convergence result we introduced a box of size p in which the distribution of the excess of mass is performed. It seems that the size of the box (parameter p) is not necessarily an important parameter if one uses a high-order prediction scheme as the Lax-Wendroff scheme for non oscillating computations. But with a more anti-dissipative prediction scheme as the Downwind scheme, the results can vary with p. When p is too large the numerical solution may not be correct. It gives some indication that it is much preferable to restrict ourselves to local Repair (*i.e* with a local redistribution of the mass).

An important feature of the Repair paradigm is the simplicity and versatility in any dimension. Moreover the repair process is independent of the kind of mesh used; cell-centered values or nodal values can be repaired the same way. We only need the notion of neighborhood to define the bounds and to redistribute the amount of conservative variable. Therefore any repair algorithm is suitable for staggered formulation where physical variables are not defined at the same place — classical Lagrangian schemes in CFD deal with cell-centered density, nodal velocity etc.
The test cases in 1D show non only the convergence of the repair process on transport equation but on 1D gas dynamics as well. We have shown on several problems that the repair process is usefull to correct some bad behaviors of Lagrange+Remap schemes for a non-linear system of PDEs.

[1]For gas dynamics, the maximum principle means that if m, μ, E are remapped and ρ, u, ε are the associated repaired variables, then no new ewtrema is created in the solution ρ, u, ε of this system. For a general system it is an open problem to design a criterion for repairing

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A A simple proof of inequality 41 in the case $\nu = \frac{1}{2}$

We prove here that there exists $C > 0$ such that for $\nu = \frac{1}{2}$ one has

$$||(I + \Delta t(T - I))^q (T - I)|| \leq C \frac{1}{\sqrt{q+1}}. \tag{50}$$

A complete proof is in given [7], that also takes into account variable time steps. One has

$$\frac{1}{2^q}(I + T)^q (T - I) = \frac{1}{2^q} \left(\sum_j \left(\begin{pmatrix} q \\ j-1 \end{pmatrix} - \begin{pmatrix} q \\ j \end{pmatrix} \right) T^j \right). \tag{51}$$

The binomial coefficient is $\begin{pmatrix} q \\ j \end{pmatrix} = \frac{q!}{j!(q-j)!}$. The series $j \mapsto \begin{pmatrix} q \\ j \end{pmatrix}$ is increasing from 0 to $[\frac{q}{2}] \pm 1$ and is decreasing from $[\frac{q}{2}] \pm 1$ to m, where $[\frac{q}{2}] \in \mathbf{N}$ is the entire part of $\frac{q}{2}$. Thus

$$\sum_j \left(\left| \begin{pmatrix} q \\ j-1 \end{pmatrix} - \begin{pmatrix} q \\ j \end{pmatrix} \right| \times \left\| T^j \right\| \right) \leq 2 \begin{pmatrix} q \\ [\frac{q}{2}] \pm 1 \end{pmatrix} \times \approx 2 \frac{q!}{(\frac{q}{2}!)^2}. \tag{52}$$

For a positive number m, the Stirling formula [1] gives that $m! \approx \sqrt{2\pi m}(\frac{m}{e})^m$. Plugging in the right hand side of our last inequality, one gets that

$$\frac{q!}{(\frac{q}{2}!)^2} \approx \frac{\sqrt{2\pi q}(\frac{q}{e})^q}{\left(\sqrt{\pi q}(\frac{q}{2e})^{\frac{q}{2}}\right)^2} \approx 2^q \sqrt{\frac{2}{\pi}} \times \frac{1}{\sqrt{q}}. \tag{53}$$

From (51-53) one gets

$$||(I + \frac{1}{2}(T - I))^q (T - I)|| \lesssim \left(2\sqrt{\frac{2}{\pi}} \right) \frac{1}{\sqrt{q+1}}.$$

It ends the proof of the lemma in the case $\nu = \frac{1}{2}$. This proof can easily be generalized for $\nu \neq \frac{1}{2}$.

References

[1] Abramowitz M. and Stegun I.A., Handbook of mathematical functions, Dover publications, New York.

[2] F. Bouchut, An antidiffusive entropy scheme for monotone scalar conservation laws, J. Sci. Comput. 21 (2004), 1-30.

[3] J.Campbell, M.Shashkov – *A compatible lagrangian hydrodynamics algorithm for unstructured grids* – Selcuk Journal of Applied Mathematics, 4, (2003) pp. 53-70; report version at `http://cnls.lanl.gov/~shashkov`

[4] E.J.Caramana, D.E.Burton, M.J.Shashkov, and P.P.Whalen – *The Construction of Compatible Hydrodynamics Algorithms Utilizing Conservation of Total Energy* – Journal of Computational Physics, 146, pp. 227–262, (1998).

[5] E.J.Caramana and M.Shashkov – *Elimination of Artificial Grid Distortion and Hourglass-Type Motions by Means of Lagrangian Subzonal Masses and Pressures* – Journal of Computational Physics, 142, pp. 521–561, (1998).

[6] B.Després and F.Lagoutière – *Contact discontinuity capturing schemes for linear advection and compressible gas dynamics* – J. Sci. Comput. 16 (2001), no. 4, 479–524 (2002).

[7] B.Després – *Lax theorem and finite volume schemes* – Math. Comp. 73 (2004), no. 247, 1203–1234.

[8] B.Després – Proceedings of the *Hispano-French Numerical Analysis School* – Cadix 2004.

[9] R.Loubère and M.Shashkov – *A subcell remapping method on Staggered Polygonal Grids for Arbitrary-Lagrangian-Eulerian methods* – Journal of Computational Physics, Volume 209, Issue 1, 10 October 2005, Pages 105-138 and report LA-UR-04-6692 (2004).

[10] R.Loubère, M.Staley and B.Wendroff – *The Repair paradigm: new algorithms and application to gas dynamics* Journal of Computational Physics, Volume 211, Issue 2, 20 January 2006, Pages 385-404 and report LA-UR-04-0795 (2004).

[11] M.Shashkov, B.Wendroff – *The Repair Paradigm and Application to Conservation Laws* – Journal of Computational Physics, 198, pp. 265-277, 2004.

[12] Z. Xu and C.-W. Shu, Anti-diffusive flux corrections for high order finite difference WENO schemes, submitted to Journal of Computational Physics.