A simple and accurate coupled HLL-type approximate Riemann solver for the two-fluid two-pressure model of compressible flows

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Abstract

This paper is concerned with the design of a very simple and efficient Godunov-type method for the so-called two-fluid two-pressure compressible model for two-phase flows. It is a contribution to the proceedings of a workshop organized by the EDF R&D French utility company and devoted to the verification of numerical schemes for two-phase flows. The present study focuses on the convective part of the two-fluid two-pressure model and is a short form of a longer paper [1] where the numerical strategy also takes into account additional terms associated with sources, pressure relaxation and drag forces. Numerical simulations and comparisons with other strategies are proposed in the last section.

1 Introduction

This paper is concerned with the numerical approximation of the solutions of the so-called two-fluid two-pressure model which was first proposed by Baer Nunziato [for granular energetic combustible materials embedded in gaseous combustion products Here, our basic motivation is the computation of compressible and subsonic two-phase flows involved in the nuclear industry We will focus on the numerical approximation of the convective part of this model and we will neglect here the pressure relaxation and drag force terms, the dissipation terms due to laminar or turbulent viscosity, the heat conduction and the external forces Actually, the method proposed in the present paper is nothing but a simplification of the Godunov-type method introduced in [1] where the full model is treated and a particular attention is paid to the asymptotic properties of the solutions when the source terms are stiff Here, all these terms are set to be zero Said differently, the content of the present paper is not new but already contained in [1] in a more complicated way since additional terms are taken into account

Mathematical and numerical studies of the model under consideration or related ones are abundant in the literature and can be found in many papers like (without any attempt to be exhaustive) Embid Baer [3, Stewart Wendroff [, Abgrall Saurel [, [1], Kapila *et al.* [3 , Glimm *et al.* [31], Abgrall Saurel [, Gavrilyuk Saurel [3 , Gallouët, Hérard Seguin [,

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Coquel, Gallouët, Hérard Seguin [1 , and more recently Ambroso, Chalons, Coquel Galié [, Tokareva Toro [6 , Ambroso *et al.* [1, Coquel *et al.* [, [6 , [1 , [, see also the references therein

One of the main features of the two-fluid two-pressure model is to consider two velocities u_1 and u_2 and two pressures p_1 and p_2 which are associated with the two phases and are not necessarily equal Unlike the so-called two-fluid one-pressure models where the two pressures are equal, namely p p_1 p_2 in the closure relations, this feature makes real (instead of complex) the characteristic speeds of the model Moreover, the model is always hyperbolic in the subsonic regime of interest here From a mathematical viewpoint, this flow regime expresses that some of the eigenvalues do not coincide and that the so-called resonance phenomenon does not occur, which turns out to be relevant in the nuclear energy industry framework which motivates this work

From a numerical point of view, the size of the model, its nonlinearities and the presence of nonconservative products make pretty difficult the design of cheap and efficient numerical schemes for approximating its solutions Again, the literature is large on this topic as briefly reported below, but let us observe from now on that most of the proposed schemes are based on nonlinear exact or approximate Riemann solvers and/or consider a specific choice for the interfacial velocity u_I involved in the governing equations Here, our first objective is to present a Godunov-type method based on an approximate Riemann solver which is explicitly defined and able to consider a continuous set of interfacial velocities The simplicity and accuracy of the method make it well-adapted to the nuclear industry

Let us now briefly review some of the existing schemes for the two-fluid two-pressure model A first group of works is due to Saurel and collaborators Let us first mention that Saurel Abgrall [and Andrianov, Saurel Warnecke [for instance (see also Saurel Lemetayer [for a multidimensional framework) take into account the nonconservative terms by means of a *free streaming* physical condition associated with uniform velocity and pressure profiles The discretization technique of [is improved by the same authors in [1 Then, in Andrianov Warnecke [6 and Schwendeman, Wahle Kapila [3, the common objective is to get exact solutions for the Riemann problem of the model The approach is *inverse* in [6 in the sense that the initial left and right states are obtained as function of the intermediate states of the solution On the contrary, a *direct* iterative approach is used in [3 leading to exact solutions of the Riemann problem for any initial left and right states See also the work of Deledicque Papalexandris [Another *direct* approach to construct theoretical solutions is proposed in Castro Toro [1 In this work the authors propose to solve the Riemann problem approximately assuming that all the nonlinear characteristic fields are associated with rarefaction waves More recently, Tokareva Toro propose in [6 a HLLC-type approximate Riemann solver which takes into account all the seven waves that are naturally present in the model, and that can be seen as a similar but faster approach in comparison to the exact solver proposed in [3 Finally, all these (approximate or exact) solutions are used to develop a Godunov-type method At last, other finite volumes methods have been used For instance in Gallouët *et al.* [(see also Guillemaud [3), the approximation of the convective terms of the system is based on the Rusanov scheme (Rusanov [ ) and the so-called VFRoe-ncv scheme (Buffard *et al.* [ ), these strategies being adapted to the nonconservative framework, at least for systems where nonconservative products are not active in genuinely nonlinear fields. In Munkejord [6] and Karni *et al.* [ ], the authors use Roe-type schemes. For additional theoretical and numerical studies devoted to the two-fluid two-pressure model, we also refer to the recent works [3 , 55 , [1 , [5 , [5 , [16 , [1 , [18 , [8 , [, [, [

The outline of the paper is classical and as follows. Section 2 presents the model under consideration. Section 3 gives the Godunov-type strategy and the underlying explicit approximate Riemann solver. At last, Section 4 is concerned with numerical experiments.

2 The model under consideration

We consider the following non conservative system of partial differential equations in one space dimension:

$$\begin{cases} \partial_t \alpha_k + u_I \partial_x \alpha_k = 0, \\ \partial_t(\alpha_k \varrho_k) + \partial_x(\alpha_k \varrho_k u_k) = 0, \\ \partial_t(\alpha_k \varrho_k u_k) + \partial_x(\alpha_k(\varrho_k u_k^2 + p_k)) - p_I \partial_x \alpha_k = 0, \\ \partial_t(\alpha_k \varrho_k e_k) + \partial_x(\alpha_k(\varrho_k e_k + p_k) u_k) - p_I u_I \partial_x \alpha_k = 0, \end{cases} \quad (1)$$

where $k = 1, 2$. In these equations, α_k, ϱ_k, u_k, e_k and p_k denote the volume fraction, density, velocity, specific total energy and pressure of the phase k, $k = 1, 2$. We assume that the phases are unmixable, which writes here

$$\alpha_1 + \alpha_2 = 1, \quad (2)$$

and obey an equation of state of the form

$$p_k = p_k(\varrho_k, \varepsilon_k), \quad k = 1, 2, \quad (3)$$

where $\varepsilon_k = e_k - u_k^2/2$ is the specific internal energy.
The nonconservative products $p_I \partial_x \alpha_k$ and $p_I u_I \partial_x \alpha_k$, where p_I and u_I have to be defined, can be understood as coupling terms between two classical gas dynamics systems of partial differential equations associated with the phases $k = 1, 2$. They will play an important role in what follows.
Let us introduce the following condensed form for (1), namely

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) + \mathbf{B}(\mathbf{U}) \partial_x \mathbf{U} = 0, \quad (4)$$

where

$$\mathbf{U} = \begin{pmatrix} \alpha_1 \\ \mathbf{U}_1 \\ \mathbf{U}_2 \end{pmatrix}, \quad \mathbf{U}_k = \begin{pmatrix} \alpha_k \varrho_k \\ \alpha_k \varrho_k u_k \\ \alpha_k \varrho_k e_k \end{pmatrix}, \quad (5)$$

$$\mathbf{F}(\mathbf{U}) = \begin{pmatrix} 0 \\ \mathbf{F}_1(\mathbf{U}_1) \\ \mathbf{F}_2(\mathbf{U}_2) \end{pmatrix}, \quad \mathbf{F}_k(\mathbf{U}_k) = \begin{pmatrix} \alpha_k \varrho_k u_k \\ \alpha_k(\varrho_k u_k^2 + p_k) \\ \alpha_k(\varrho_k e_k + p_k) u_k \end{pmatrix}, \quad (6)$$

$k, l = 1, \ , l \neq k$ and

$$
\mathbf{B}(\mathbf{U}) = \begin{pmatrix} u_I & \\ & \\ -p_I & \\ -p_I u_I & \mathbf{O} \\ & \\ p_I & \\ p_I u_I & \end{pmatrix}. \qquad (\ )
$$

Before defining the interfacial velocity and pressure u_I and p_I, let us first recall the basic hyperbolicity properties of (1) (see for instance [for more details) First of all, easy calculations show that the eigenvalues of the Jacobian matrix $\mathbf{F}'(\mathbf{U})+\mathbf{B}(\mathbf{U})$ are real and given by u_I, u_k, $u_k \pm c_k$, $k = 1, \ ,$ where c_k denotes the sound speed of the phase k However, the system (1) is only weakly hyperbolic since hyperbolicity can be lost when resonance occurs, that is to say when $u_I = u_k \pm c_k$ for some k Moreover and away from resonance, the characteristic fields associated with $u_k \pm c_k$ are genuinely nonlinear and the characteristic field associated with u_k is linearly degenerate At last, the characteristic field associated with u_I is linearly degenerate provided that

$$
u_I = \beta u_1 + (1-\beta) u_2 \quad \text{with} \quad \beta = \frac{\chi \alpha_1 \varrho_1}{\chi \alpha_1 \varrho_1 + (1-\chi)\alpha_2 \varrho_2} \qquad (8)
$$

where $\chi \in [\ , 1$ is a constant We will adopt this definition with $\chi = \ , 5 \ , 1$ in the numerical experiments but any different value could be considered as well The interfacial pressure p_I will be defined by

$$
p_I = \mu p_1 + (1-\mu) p_2 \quad \text{with} \quad \mu = \mu(\mathbf{U}) \in [\ , 1 . \qquad (\ )
$$

The following definition of μ, namely

$$
\mu = \mu(\beta) = \frac{(1-\beta) T_2}{\beta T_1 + (1-\beta) T_2} \qquad (1\ )
$$

where T_k is the temperature of the phase k, is motivated by entropy considerations More precisely, given a monotonically decreasing and convex C^1 function $\phi = \phi(s)$, it can be proved that the couple (η, q) with

$$
\eta = \eta(\mathbf{U}) = \sum_{k=1}^{2} \alpha_k \varrho_k \phi(s_k), \quad q = q(\mathbf{U}) = \sum_{k=1}^{2} \alpha_k \varrho_k \phi(s_k) u_k \qquad (11)
$$

defines a natural entropy-entropy flux pair in order to select the shock solutions of (1) In other words, we impose the entropy inequality

$$
\frac{\partial \eta}{\partial t} + \frac{\partial q}{\partial x} \leq \qquad (1\ )
$$

in the distributional sense for a shock discontinuity to be admissible As far as discontinuities associated with linearly degenerate characteristic fields are concerned, they are naturally parametrized thanks to a set of Riemann invariants We refer again to [for more details in the cases $\chi = \ , 5 \ , 1$

3 The numerical method

In this section, we describe a Godunov-type method based on the design of a relevant approximate Riemann solver for the nonconservative system ( ). We will pay a particular attention to the discretization of the coupling wave between the two phases, which means here to the correct approximation of the nonconservative products of the model.

We first introduce a constant space step Δx and a constant time step Δt, and define the cell centers x_i and the intermediate times t^n by

$$x_i = i\Delta x,\ i \in \mathbb{Z}, \quad t^n = n\Delta t,\ n \in \mathbb{N}.$$

We also set $\Delta = (\Delta x, \Delta t)$ and denote by $\mathbf{U}_i^n$ the approximate value of $\mathbf{U}(x_i, t_n)$ and we set $\mathbf{U}_i^0 = \mathbf{U}_0(x_i)$ for all i where $\mathbf{U}_0$ denotes the initial condition. Then, starting from $\mathbf{U}^n = (\mathbf{U}_i^n)_{i\in\mathbb{Z}}$, it is a matter of defining the approximate solution $\mathbf{U}^{n+1}$ at time t^{n+1}.

Following Gallice [6], [ ] and [8], we then briefly recall the notion of consistency of a simple Riemann solver of the form

$$\mathbf{W}_\Delta(x/t; \mathbf{U}_L, \mathbf{U}_R) = \begin{cases} \mathbf{U}_1 = \mathbf{U}_L, & x/t < \sigma_1, \\ \mathbf{U}_k, & \sigma_{k-1} < x/t < \sigma_k, \quad k = 2, .., m, \\ \mathbf{U}_{m+1} = \mathbf{U}_R, & x/t > \sigma_m. \end{cases} \tag{13}$$

Generically, such a solver is made of m waves with speeds $\sigma_k = \sigma_k(\mathbf{U}_L, \mathbf{U}_R)$, $1 \leq k \leq m$, and $m-1$ intermediate states $\mathbf{U}_k$, $2 \leq k \leq m$ such that

$$\lim_{\substack{\mathbf{U}_L, \mathbf{U}_R \to \mathbf{U} \\ \Delta \to 0}} \mathbf{W}_\Delta(x/t; \mathbf{U}_L, \mathbf{U}_R) = \mathbf{U}. \tag{14}$$

Under the CFL condition

$$\max_{1\leq k\leq m} |\sigma_k| \frac{\Delta t}{\Delta x} \leq \frac{1}{2}, \tag{15}$$

the Riemann solver (13) is said to be consistent with ( ) if the following relation holds true:

$$\Delta \mathbf{F} + \mathbf{B}_\Delta(\mathbf{U}_L, \mathbf{U}_R)\Delta \mathbf{U} = \sum_{k=1}^{m} \sigma_k(\mathbf{U}_{k+1} - \mathbf{U}_k), \tag{16}$$

where $\Delta \mathbf{U} = \mathbf{U}_R - \mathbf{U}_L$, $\Delta \mathbf{F} = \mathbf{F}(\mathbf{U}_R) - \mathbf{F}(\mathbf{U}_L)$ and $\mathbf{B}_\Delta(\mathbf{U}_L, \mathbf{U}_R)$ is a $p \times p$ matrix with

$$\lim_{\substack{\mathbf{U}_L, \mathbf{U}_R \to \mathbf{U} \\ \Delta \to 0}} \mathbf{B}_\Delta(\mathbf{U}_L, \mathbf{U}_R) = \mathbf{B}(\mathbf{U}). \tag{17}$$

Note that (16) allows to recover the usual definition of consistency for systems of conservation laws when $\mathbf{B} = \mathbf{B}_\Delta = \mathbf{O}$.

Once such an approximate Riemann solver is designed, the Godunov-type method is defined by

$$\Delta x \mathbf{U}_i^{n+1} = \int_{-\frac{\Delta x}{2}}^{0} \mathbf{W}_\Delta(\frac{x}{\Delta t}; \mathbf{U}_i^n, \mathbf{U}_{i+1}^n)dx + \int_{0}^{\frac{\Delta x}{2}} \mathbf{W}_\Delta(\frac{x}{\Delta t}; \mathbf{U}_{i-1}^n, \mathbf{U}_i^n)dx, \quad (8)$$

which equivalently writes

$$\begin{aligned} \mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}(\mathbf{G}_{i+\frac{1}{2}}^n - \mathbf{G}_{i-\frac{1}{2}}^n) - \frac{\Delta t}{\Delta x}\Big\{\mathbf{B}_{i-\frac{1}{2}}^n(\mathbf{U}_i^n - \mathbf{U}_{i-1}^n) + \\ + \mathbf{B}_{i+\frac{1}{2}}^n(\mathbf{U}_{i+1}^n - \mathbf{U}_i^n)\Big\} \end{aligned} \quad (9)$$

where

$$\mathbf{G}_\Delta(\mathbf{U}_L, \mathbf{U}_R) = \frac{1}{2}\Big(\mathbf{F}(\mathbf{U}_L) + \mathbf{F}(\mathbf{U}_R) - \sum_{k=1}^{m} |\sigma_k|(\mathbf{U}_{k+1} - \mathbf{U}_k)\Big) \quad (10)$$

and for all i

$$\mathbf{G}_{i+\frac{1}{2}}^n = \mathbf{G}_\Delta(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n), \ \mathbf{B}_{i+\frac{1}{2}}^n = \mathbf{B}_\Delta(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n). \quad (11)$$

The proposed approximate Riemann solver. In order to design the approximate Riemann solver $\mathbf{W}_\Delta$, we suggest the following form

$$\mathbf{W}_\Delta = \begin{pmatrix} \alpha_{1,\Delta} \\ \mathbf{W}_{1,\Delta} \\ \mathbf{W}_{2,\Delta} \end{pmatrix}(x/t; \mathbf{U}_L, \mathbf{U}_R) \quad (12)$$

with

$$\mathbf{W}_{k,\Delta}(x/t; \mathbf{U}_L, \mathbf{U}_R) = \begin{cases} \mathbf{U}_{k,L}, & x/t < \sigma_{k,1}, \\ \mathbf{U}_{k,L}^*, & \sigma_{k,1} < x/t < \sigma_2, \\ \mathbf{U}_{k,R}^*, & \sigma_2 < x/t < \sigma_{k,3}, \\ \mathbf{U}_{k,R}, & x/t > \sigma_{k,3}, \end{cases} \quad (13)$$

and

$$\alpha_{k,\Delta}(x/t; \mathbf{U}_L, \mathbf{U}_R) = \begin{cases} \alpha_{k,L}, & x/t < \sigma_2, \\ \alpha_{k,R}, & x/t > \sigma_2, \end{cases} \quad (14)$$

where the wave speed estimates are taken to be

$$\begin{cases} \sigma_{k,1} = u_{k,L} - C_{k,L}/\varrho_{k,L}, \\ \sigma_{k,2} = \sigma_2 = u_I^*, \\ \sigma_{k,3} = u_{k,R} + C_{k,R}/\varrho_{k,R}. \end{cases} \quad (15)$$

Here, $\sigma_{k,1}$ and $\sigma_{k,3}$ are linearizations of the acoustic waves associated with each phase (the constants $C_{k,L}$ and $C_{k,R}$ play the role of Lagrangian sound speeds) and u_I^* is an approximation of the speed of propagation of the coupling wave. Such an approximate solver can be understood as two coupled HLL solvers (see [3]) for each phase. Note that throughout the paper, we will consider subsonic flows leading to the following a priori given wave configuration,

$$\sigma_{k,1} < u_I^* < \sigma_{k,3}.$$

In order to define u_I^* and the intermediate states, let us first observe that the consistency relation (16) can be written as follows,

$$\left\{ \begin{array}{l} \Delta \mathbf{F}_k - \begin{pmatrix} \tilde{p}_I \\ \tilde{p}_I u_I^* \end{pmatrix} \Delta \alpha_k = \sigma_{k,1}(\mathbf{U}_{k,L}^* - \mathbf{U}_{k,L}) + \\ \\ +u_I^*(\mathbf{U}_{k,R}^* - \mathbf{U}_{k,L}^*) + \sigma_{k,3}(\mathbf{U}_{k,R} - \mathbf{U}_{k,R}^*). \end{array} \right. \tag{26}$$

where $\tilde{p}_I$ is a consistent approximation of the interfacial pressure. Note that (26) implicitly defines in passing the matrix $\mathbf{B}_\Delta$. These relations must be valid for the sake of consistency, but are not sufficient to single out the intermediates states. With this in mind, we first impose the Rankine-Hugoniot jump relations for the mass conservation equation at each wave with speed $\sigma_{k,l}$, $1 \leq l \leq 3$

$$\left\{ \begin{array}{l} \sigma_{k,1}(\varrho_{k,L}^* - \varrho_{k,L}) = \varrho_{k,L}^* u_{k,L}^* - \varrho_{k,L} u_{k,L}, \\ \sigma_2(\alpha_{k,R}\varrho_{k,R}^* - \alpha_{k,L}\varrho_{k,L}^*) = \alpha_{k,R}\varrho_{k,R}^* u_{k,R}^* - \alpha_{k,L}\varrho_{k,L}^* u_{k,L}^*, \\ \sigma_{k,3}(\varrho_{k,R} - \varrho_{k,R}^*) = \varrho_{k,R} u_{k,R} - \varrho_{k,R}^* u_{k,R}^*. \end{array} \right. \tag{27}$$

Note that summing these equations gives the first component of the consistency condition (26). Note also that (27) yields

$$\left\{ \begin{array}{l} \sigma_{k,1} = u_{k,L} - C_{k,L}/\varrho_{k,L} = u_{k,L}^* - C_{k,L}/\varrho_{k,L}^*, \\ \sigma_{k,3} = u_{k,R} + C_{k,R}/\varrho_{k,R} = u_{k,R}^* + C_{k,R}/\varrho_{k,R}^*, \end{array} \right. \tag{28}$$

and

$$j_k \stackrel{def}{=} \alpha_{k,L}\varrho_{k,L}^*(u_{k,L}^* - u_I^*) = \alpha_{k,R}\varrho_{k,R}^*(u_{k,R}^* - u_I^*). \tag{29}$$

The momentum and energy equations will be treated in a slightly different way since they contain the nonconservative products. We suggest to take these nonconservative products into account across the coupling wave associated with $\sigma_2 = u_I^*$. More precisely, we impose the Rankine-Hugoniot jump relations for the homogeneous momentum conservation equation across the acoustic waves $\sigma_{k,1}$ and $\sigma_{k,3}$,

$$\left\{ \begin{array}{l} \sigma_{k,1}\alpha_{k,L}(\varrho_{k,L}^* u_{k,L}^* - \varrho_{k,L} u_{k,L}) = \alpha_{k,L}(\varrho_{k,L}^* u_{k,L}^{*2} - \varrho_{k,L} u_{k,L}^2) + \Pi_{k,L}^* - \Pi_{k,L}, \\ \sigma_{k,3}\alpha_{k,R}(\varrho_{k,R} u_{k,R} - \varrho_{k,R}^* u_{k,R}^*) = \alpha_{k,R}(\varrho_{k,R} u_{k,R}^2 - \varrho_{k,R}^* u_{k,R}^{*2}) + \Pi_{k,R} - \Pi_{k,R}^*, \end{array} \right. \tag{30}$$

and the generalized Rankine-Hugoniot relation

$$\left\{ \begin{array}{l} u_I^*(\alpha_{k,R}\varrho_{k,R}^* u_{k,R}^* - \alpha_{k,L}\varrho_{k,L}^* u_{k,L}^*) = \alpha_{k,R}\varrho_{k,R}^* u_{k,R}^{*2} - \alpha_{k,L}\varrho_{k,L}^* u_{k,L}^{*2} - \\ \qquad -\tilde{p}_I(\alpha_{k,R} - \alpha_{k,L}) + \Pi_{k,R}^* - \Pi_{k,L}^* \end{array} \right. \tag{31}$$

across σ_2. Again, (30) and (31) imply the validity of the second component of the consistency relation (26), namely

$$\left\{ \begin{array}{l} \Delta(\alpha_k \varrho_k u_k^2 + \Pi_k) - \tilde{p}_I \Delta \alpha_k = \sigma_{k,1}\alpha_{k,L}(\varrho_{k,L}^* u_{k,L}^* - \varrho_{k,L} u_{k,L}) + \\ +u_I^*(\alpha_{k,R}\varrho_{k,R}^* u_{k,R}^* - \alpha_{k,L}\varrho_{k,L}^* u_{k,L}^*) + \sigma_{k,3}\alpha_{k,R}(\varrho_{k,R} u_{k,R} - \varrho_{k,R}^* u_{k,R}^*). \end{array} \right.$$

Note that (30) involves some pressure linearization terms $\Pi_{k,L}^*$ and $\Pi_{k,R}^*$ which are unknown at this stage and have to be defined.

Let us now define the approximate interfacial velocity u_I^*. The definition (8) of u_I leads us to set

$$u_I^* = \beta_L u_{1,L}^* + (1-\beta_L)u_{2,L}^*, \quad \beta_L = \frac{\chi\alpha_{1,L}\varrho_{1,L}^*}{\chi\alpha_{1,L}\varrho_{1,L}^* + (1-\chi)\alpha_{2,L}\varrho_{2,L}^*}$$

or equivalently

$$u_I^* = \beta_R u_{1,R}^* + (1-\beta_R)u_{2,R}^*, \quad \beta_R = \frac{\chi\alpha_{1,R}\varrho_{1,R}^*}{\chi\alpha_{1,R}\varrho_{1,R}^* + (1-\chi)\alpha_{2,R}\varrho_{2,R}^*}.$$

This amounts to set

$$\chi j_1 + (1-\chi)j_2 = 0. \tag{32}$$

Calculation of $(\varrho_{k,L}^*, \varrho_{k,R}^*)$, $(u_{k,L}^*, u_{k,R}^*)$, $(\Pi_{k,L}^*, \Pi_{k,R}^*)$ for $k = 1, 2$, and u_I^*. By simple but tedious calculations, the previous set of linear relations allow to obtain the following formulas

$$\begin{cases} u_{k,L}^* = \dfrac{(C_{k,L} - \varrho_{k,L}u_{k,L})j_k + \alpha_{k,L}C_{k,L}\varrho_{k,L}u_I^*}{\varrho_{k,L}(\alpha_{k,L}C_{k,L} - j_k)}, \\ \\ u_{k,R}^* = \dfrac{(C_{k,R} + \varrho_{k,R}u_{k,R})j_k + \alpha_{k,R}C_{k,R}\varrho_{k,R}u_I^*}{\varrho_{k,R}(\alpha_{k,R}C_{k,R} + j_k)}, \end{cases} \tag{33}$$

$$\begin{cases} u_I^* = \dfrac{1}{((\alpha_1 C_1)_a + (\alpha_2 C_2)_a)} \{(\sigma_{1,1} - \sigma_{1,3})j_1 + (\sigma_{2,1} - \sigma_{2,3})j_2 + \\ \\ \quad + ((\alpha_1 C_1 u_1)_a + (\alpha_2 C_2 u_2)_a) - \Delta p\}, \end{cases} \tag{34}$$

$$\begin{cases} \dfrac{1}{\varrho_{k,L}^*} = \dfrac{1}{\varrho_{k,L}} + \dfrac{u_{k,L}^* - u_{k,L}}{C_{k,L}}, \\ \dfrac{1}{\varrho_{k,R}^*} = \dfrac{1}{\varrho_{k,R}} - \dfrac{u_{k,R}^* - u_{k,R}}{C_{k,R}}, \end{cases}$$

and

$$\begin{cases} \Pi_{k,L}^* = \Pi_{k,L} - \alpha_{k,L}C_{k,L}(u_{k,L}^* - u_{k,L}), \\ \Pi_{k,R}^* = \Pi_{k,R} + \alpha_{k,R}C_{k,R}(u_{k,R}^* - u_{k,R}). \end{cases} \tag{35}$$

In these formulas and in all the sequel, $\varphi_a = \frac{1}{2}(\varphi_L + \varphi_R)$ will denote the arithmetic average of any pair of quantities (φ_L, φ_R).

Calculation of $(e_{k,L}^*, e_{k,R}^*)$ for $k = 1, 2$. First of all, the third component of the consistency relation (26), which must be valid, reads

$$\begin{aligned} &\Delta((\alpha_k\varrho_k e_k + \Pi_k)u_k) - \tilde{p}_I u_I^* \Delta\alpha_k = \sigma_{k,1}\alpha_{k,L}(\varrho_{k,L}^* e_{k,L}^* - \varrho_{k,L}e_{k,L}) + \\ &+ u_I^*(\alpha_{k,R}\varrho_{k,R}^* e_{k,R}^* - \alpha_{k,L}\varrho_{k,L}^* e_{k,L}^*) + \sigma_{k,3}\alpha_{k,R}(\varrho_{k,R}e_{k,R} - \varrho_{k,R}^* e_{k,R}^*), \end{aligned} \tag{36}$$

or equivalently

$$\Delta(\Pi_k u_k) - \tilde{p}_I u_I^* \Delta\alpha_k = (j_k - \alpha_{k,L}C_{k,L})e_{k,L}^* - (j_k + \alpha_{k,R}C_{k,R})e_{k,R}^* + 2(\alpha_k C_k e_k)_a, \tag{37}$$

which gives one equation for each phase. A second equation is needed for each phase and we proceed as follows. Like the momentum equation we are tempted to write the Rankine-Hugoniot jump relations for the homogeneous energy conservation equation at the acoustic waves. This is excluded since one would obtain three conditions for two unknowns of each phase. Let us nevertheless write these jump relations where we have replaced $e^*_{k,l}$ and $e^*_{k,R}$ by $\bar{e}_{k,l}$ and $\bar{e}_{k,R}$ respectively, we find

$$\sigma_{k,1}\alpha_{k,L}(\varrho^*_{k,L}\bar{e}_{k,L} - \varrho_{k,L}e_{k,L}) =$$
$$\alpha_{k,L}(\varrho^*_{k,L}\bar{e}_{k,L}u^*_{k,L} - \varrho_{k,L}e_{k,L}u_{k,L}) + \Pi^*_{k,L}u^*_{k,L} - \Pi_{k,L}u_{k,L}$$

and

$$\sigma_{k,3}\alpha_{k,R}(\varrho_{k,R}e_{k,R} - \varrho^*_{k,R}\bar{e}_{k,R}) =$$
$$\alpha_{k,R}(\varrho_{k,R}e_{k,R}u_{k,R} - \varrho^*_{k,R}\bar{e}_{k,R}u^*_{k,R}) + \Pi_{k,R}u_{k,R} - \Pi^*_{k,R}u^*_{k,R}.$$

Using again (8), the above relations become

$$\begin{cases} \bar{e}_{k,L} = e_{k,L} + \dfrac{1}{\alpha_{k,L}C_{k,L}}(\Pi_{k,L}u_{k,L} - \Pi^*_{k,L}u^*_{k,L}), \\ \bar{e}_{k,R} = e_{k,R} + \dfrac{1}{\alpha_{k,R}C_{k,R}}(\Pi^*_{k,R}u^*_{k,R} - \Pi_{k,R}u_{k,R}). \end{cases} \quad (38)$$

In order to determine $e^*_{k,L}$ and $e^*_{k,R}$, we suggest to solve an optimization problem. Since the consistency relation (3) is of the form

$$a_1 e^*_{k,L} - a_2 e^*_{k,R} = b$$

with $a_1 = j_k - \alpha_{k,L}C_{k,L}$, $a_2 = j_k + \alpha_{k,R}C_{k,R}$ and $b = \Delta(\Pi_k u_k) - \tilde{p}_I u^*_I \Delta\alpha_k - (\alpha_k C_k e_k)_a$, we minimize the quadratic functional

$$J(x,y) = (x - \bar{e}_{k,L})^2 + (y - \bar{e}_{k,R})^2$$

under the linear constraint

$$a_1 x - a_2 y = b_1.$$

Clearly, this optimization problem has a unique solution $(x = e^*_{k,L}, y = e^*_{k,R})$ given by

$$\begin{cases} e^*_{k,L} = \bar{e}_{k,L} + \dfrac{a_1}{a_1^2 + a_2^2}(b - a_1\bar{e}_{k,L} + a_2\bar{e}_{k,R}), \\ e^*_{k,R} = \bar{e}_{k,R} - \dfrac{a_2}{a_1^2 + a_2^2}(b - a_1\bar{e}_{k,L} + a_2\bar{e}_{k,R}). \end{cases}$$

The Riemann solver is therefore completely defined provided that we give $\tilde{p}_I$. We simply set

$$\tilde{p}_I = \tilde{\mu}(p_1)_a + (1 - \tilde{\mu})(p_2)_a \quad \text{with} \quad \tilde{\mu} = (1 - \chi)(\alpha_2\rho_2 T_2)_a. \quad (3)$$

We have thus explicitly defined a simple approximate Riemann solver which is able to deal with any interfacial velocity u_I ($\chi \in [\ ,1]$) and which imposes the validity of classical as well as generalized Rankine-Hugoniot relations across the extreme and the coupling waves. These relations amount to impose the continuity of some Riemann invariants across the discontinuities and will permit in the next section to compute exactly some isolated coupling waves when the pressure and velocity profiles are constant.

To conclude this section, let us mention that a key point for the efficiency and stability of the scheme is that of the choice of the parameters $C_{k,L}$ and $C_{k,R}$. We refer the reader to [1] for more details but notice that for the sake of simplicity, we have chosen in practice $C_{k,L} = C_{k,R}$, $k = 1,\ .$

4 Numerical results

We consider three Riemann problems and compare the numerical solutions provided by our scheme with the exact solutions and the approximate solutions given by the HLL-type method by Saurel Abgrall [ ], the strategy presented in Andrianov, Saurel Warnecke [ ], the VFRoe scheme derived by Gallouët, Hérard Seguin [ ], and the exact Godunov scheme of Schwendeman, Wahle Kapila [3]. The computational domain is [, 1] and the equations of state are $p_k = (\gamma_k - 1)\rho_k\varepsilon_k$, $k = 1,$ with $\gamma = \gamma_1 = \gamma_2 = 1.$ The initial discontinuity is located at point $x = 5$ and the left and right states $\mathbf{U}_L$ and $\mathbf{U}_R$ are defined using the primitive variables α_1, ρ_k, u_k and p_k, $k = 1,$

Test 1 : isolated coupling wave. We consider in this paragraph two isolated coupling waves propagating with velocity u_I. The first one (Test 1a) is taken from [ ] with $\chi = 5$, which gives

$$u_I = \frac{\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2}{\alpha_1\rho_1 + \alpha_2\rho_2},$$

and

$$\begin{array}{lll} \alpha_{1,L} = . , & (\rho_1, u_1, p_1)_L = (1, 1 \quad , 1 \quad ^5) & (\rho_2, u_2, p_2)_L = (1, 1 \quad , 1 \quad ^5) \\ \alpha_{1,R} = 5 , & (\rho_1, u_1, p_1)_R = (.15 , 1 \quad , 1 \quad ^5) & (\rho_2, u_2, p_2)_R = (.15 , 1 \quad , 1 \quad ^5). \end{array}$$

Solutions are given on Fig 1 and compared with the VFRoe scheme [ ] Observe that the constant velocity and pressure profiles are strictly preserved (by construction), while the void fraction and density profiles present exactly the same numerical diffusion for both schemes Fig shows the solutions with finer meshes

Test 1b takes $\chi = 1$ so that $u_I = u_1$ Initial conditions are taken from [6] and given by

$$\begin{array}{lll} \alpha_{1,L} = 8, & (\rho_1, u_1, p_1)_L = (, .35) & (\rho_2, u_2, p_2)_L = (1, , 1) \\ \alpha_{1,R} = .3, & (\rho_1, u_1, p_1)_R = (, .3, 1 \; 56) & (\rho_2, u_2, p_2)_R = (.1 \; 1, 8 \; 11, .1). \end{array}$$

Observe that the pressures and velocity u_2 are not equal anymore Solutions are presented on Fig 3 The results agree with the exact solutions and the numerical solutions given in [6] The small amplitude oscillations on ρ_1 and u_1 are due to the initial pressures and velocity u_2 desequilibrium

Test 2 : a general Riemann problem. We consider

$$\begin{array}{lll} \alpha_{1,L} = 8, & (\rho_1, u_1, p_1)_L = (1, , 1) & (\rho_2, u_2, p_2)_L = (. , , .3) \\ \alpha_{1,R} = .3, & (\rho_1, u_1, p_1)_R = (1, , 1) & (\rho_2, u_2, p_2)_R = (1, , 1). \end{array}$$

so that the exact solution contains shock, contact discontinuity and rarefaction waves, in addition to the coupling wave $u_I = u_1$ This test case is taken from [3] and the final time is $t = .$ Fig gives the results with , and 8 cells We observe that the phase presents a good agreement with the exact solution while the phase 1 suffers from overshoots and undershoots at the extreme waves Fig 5 details our results, while 6 enables to compare our solutions with the solutions given by the schemes proposed in Saurel and Abgrall [ ] (referred to

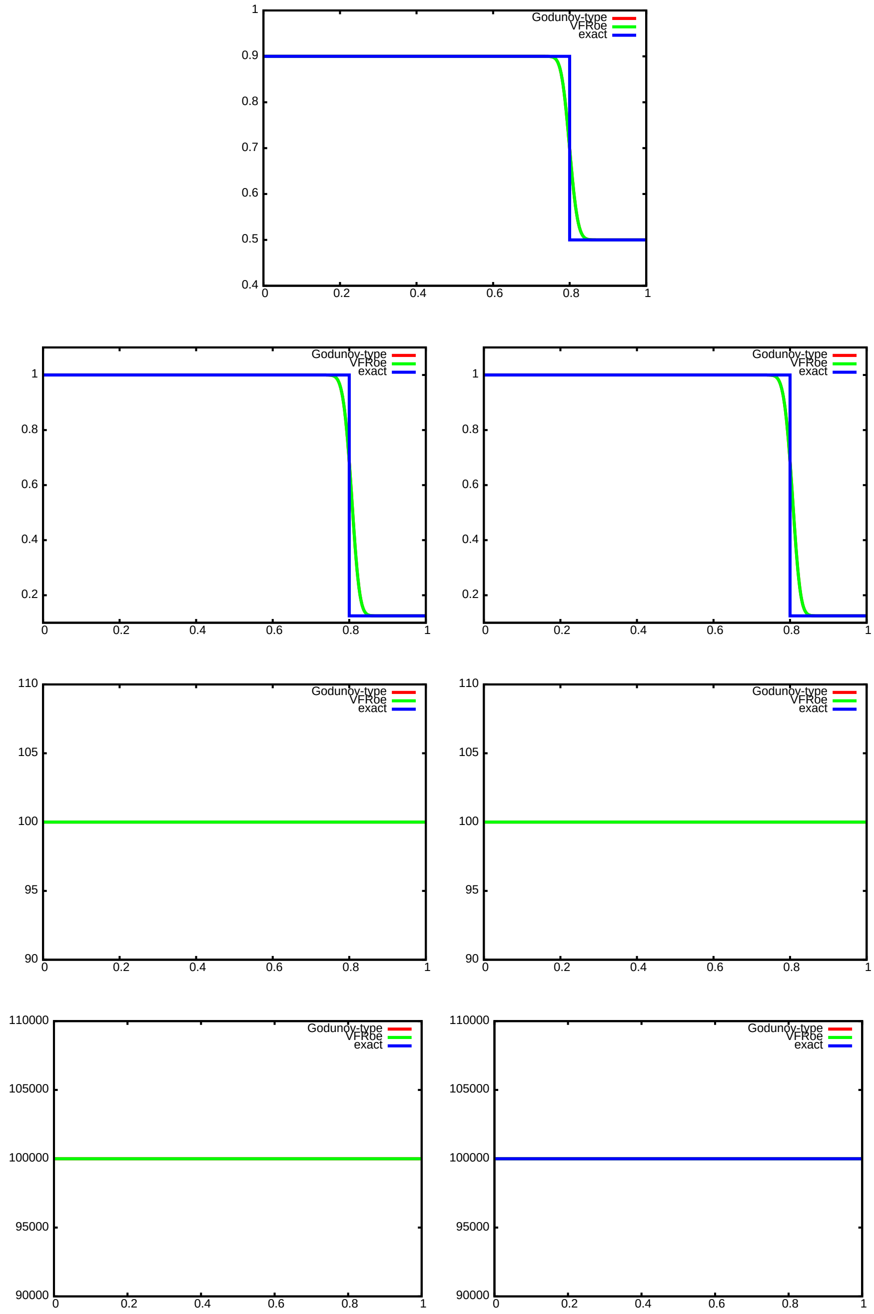

Figure 1 Comparison between exact and numerical solutions of Test 1a at time t 3 and for a 1 -point mesh From the top left to the bottom right x versus α_1, ρ_1, ρ_2, u_1, u_2, p_1, p_2

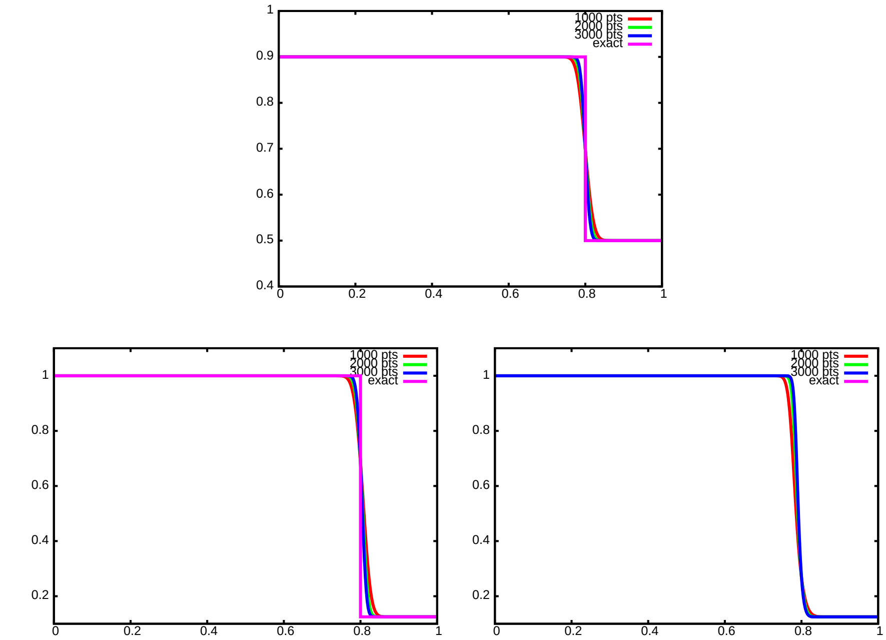

Figure Comparison between exact and numerical solutions of Test 1a at time t 3 and for several mesh sizes From the top left to the bottom right x versus α_1, ρ_1, ρ_2

as G_{HLL}), Andrianov et al [(referred to as G_{ASW}) and Schwendeman et al [3 (referred to as G_1) Recall that the latter is based on the exact resolution of the Riemann problem The intermediate states are thus perfectly captured with this method We also observe that the G_{HLL} method is the most diffusive on this test case The G_{ASW} method is less diffusive but the constant states of the u_I coupling wave are not properly captured On the contrary, our scheme behaves very well near the coupling wave and the left and right states are better evaluated

Test 3 : a general Riemann problem with several values of χ. We now compare the solutions given by several values of χ (with the same initial data as in Test), namely χ , χ 5 and χ 1 Fig shows that unlike with χ 1, the solutions obtained with χ and χ 5 do not exhibit undershoots and overshoots near the acoustic waves Interestingly, observe that the three values of χ give very different solutions which highlights the interest in a scheme capable of dealing with different values of χ

5 Conclusion and perspectives

To conclude, this paper presents a Godunov-type method to approximate the solutions of the two-fluid two-pressure diphasic model We have considered only the convective terms, while the original paper [1 also takes into account different

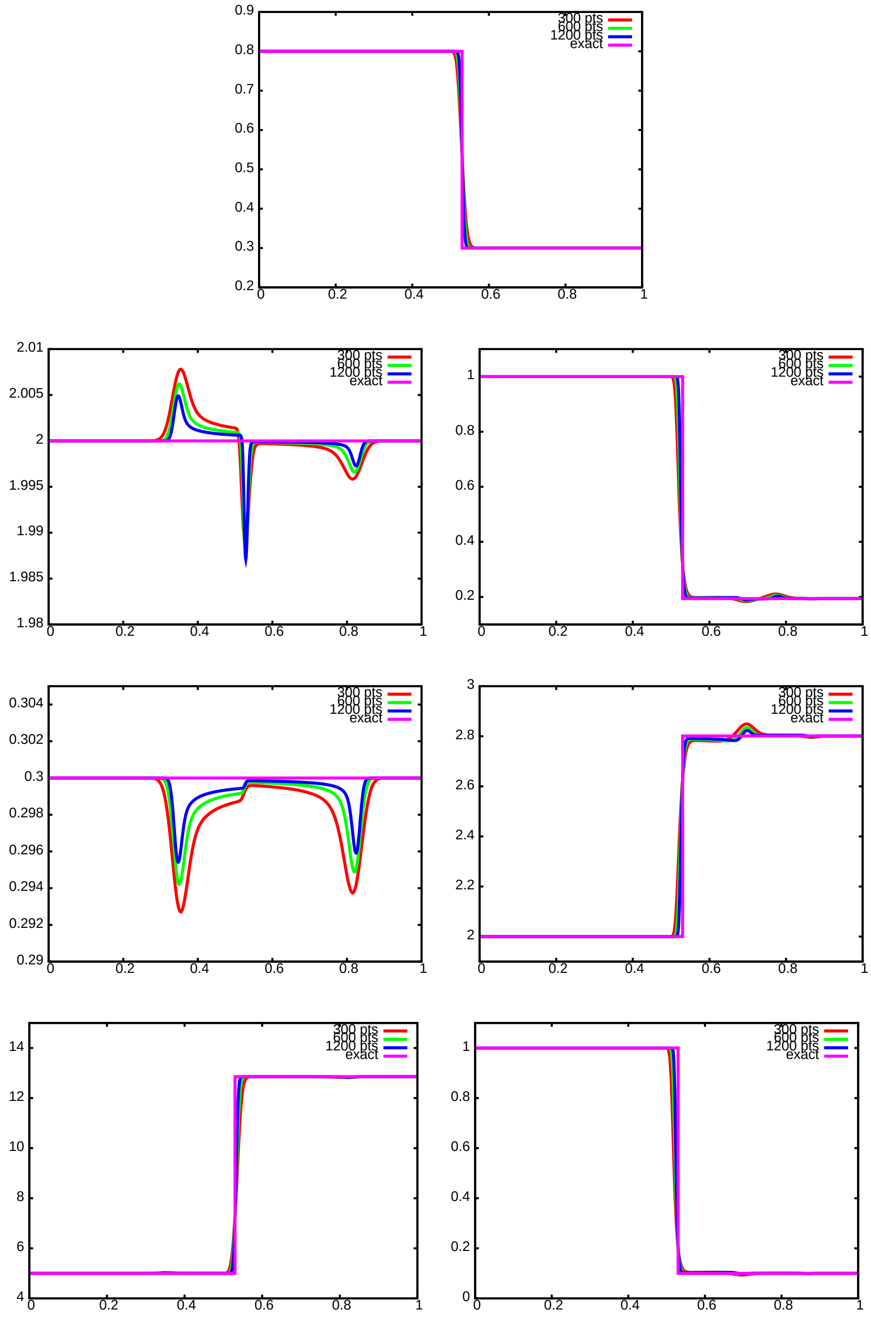

Figure 3 Comparison between exact and numerical solutions of Test 1b at time $t = 0.1$ and for several mesh sizes. From the top left to the bottom right: x versus α_1, ρ_1, ρ_2, u_1, u_2, p_1, p_2

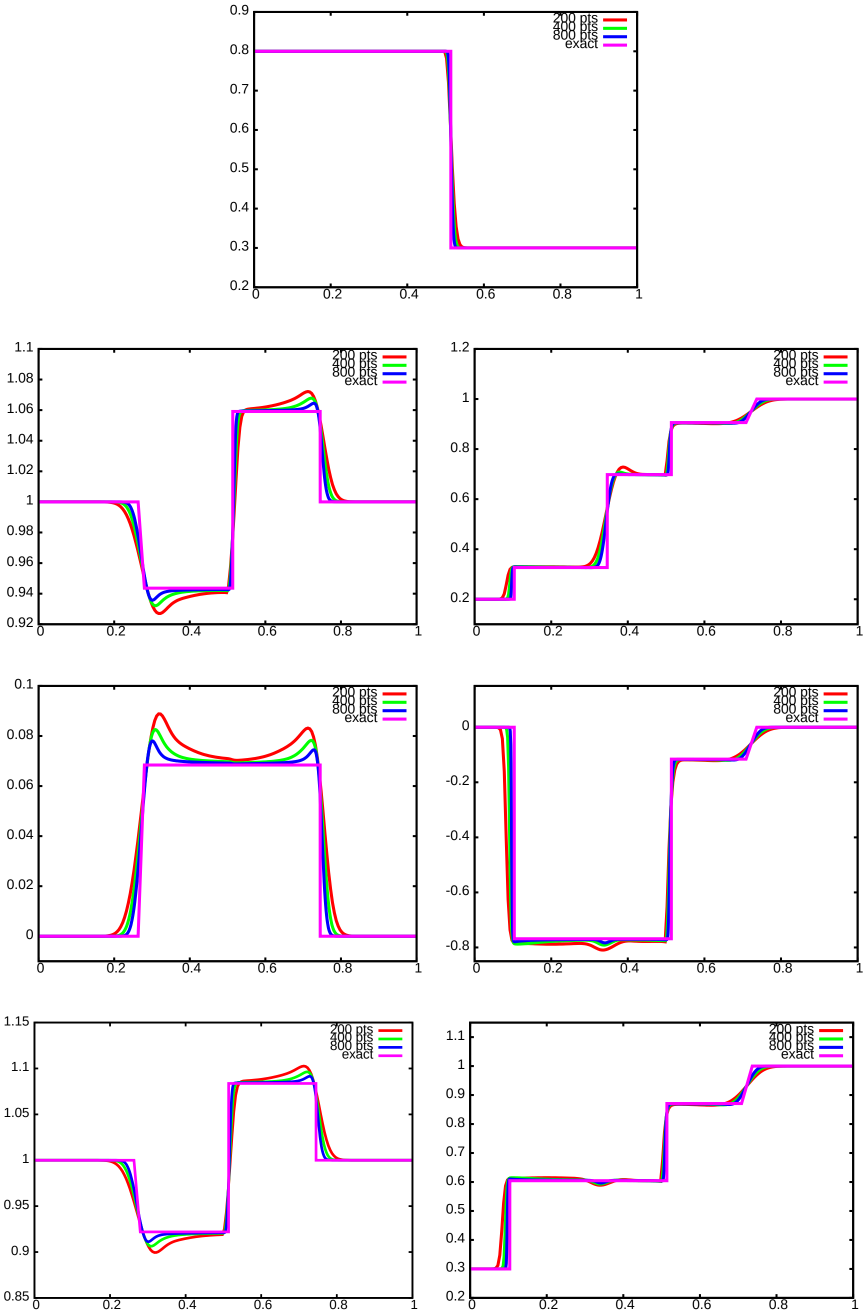

Figure   Comparison between exact and numerical solutions given by our scheme of Test   at time t . and for several mesh sizes From the top left to the bottom right x versus α_1, ρ_1, ρ_2, u_1, u_2, p_1, p_2

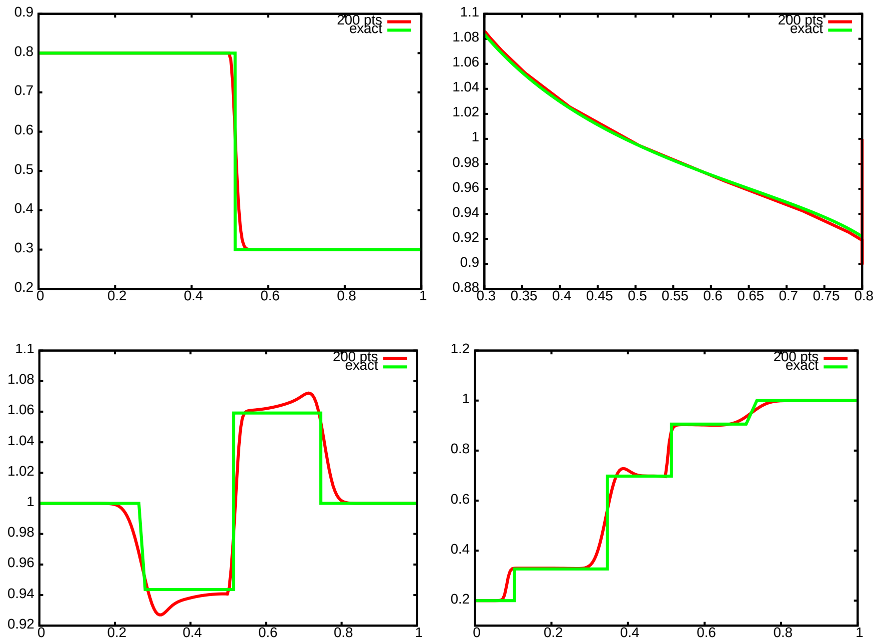

Figure 5. Behaviour of $\alpha_1(x)$, $p_1(\alpha_1)$, $\rho_1(x)$, $\rho_2(x)$ with our scheme at time t . and for Test

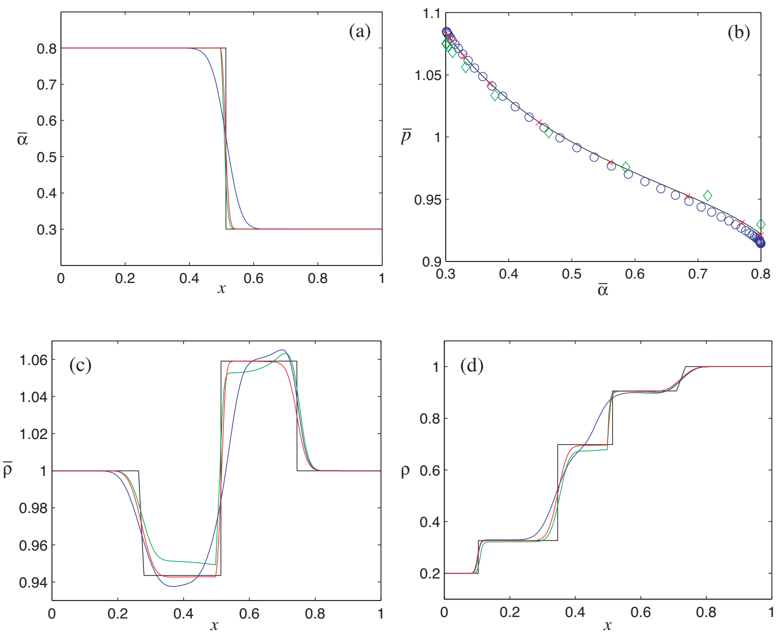

Figure 6. Behaviour of $\alpha_1(x)$, $p_1(\alpha_1)$, $\rho_1(x)$, $\rho_2(x)$ (respectively denoted by $\overline{\alpha}$, $\overline{p}$, $\overline{\rho}$ and ρ on the picture) at time t . and for Test These pictures are taken from [3] by courtesy of the authors The curves are given by method G_{HLL} (blue), G_{ASW} (green) and G_1 (red) The exact solution appears in black

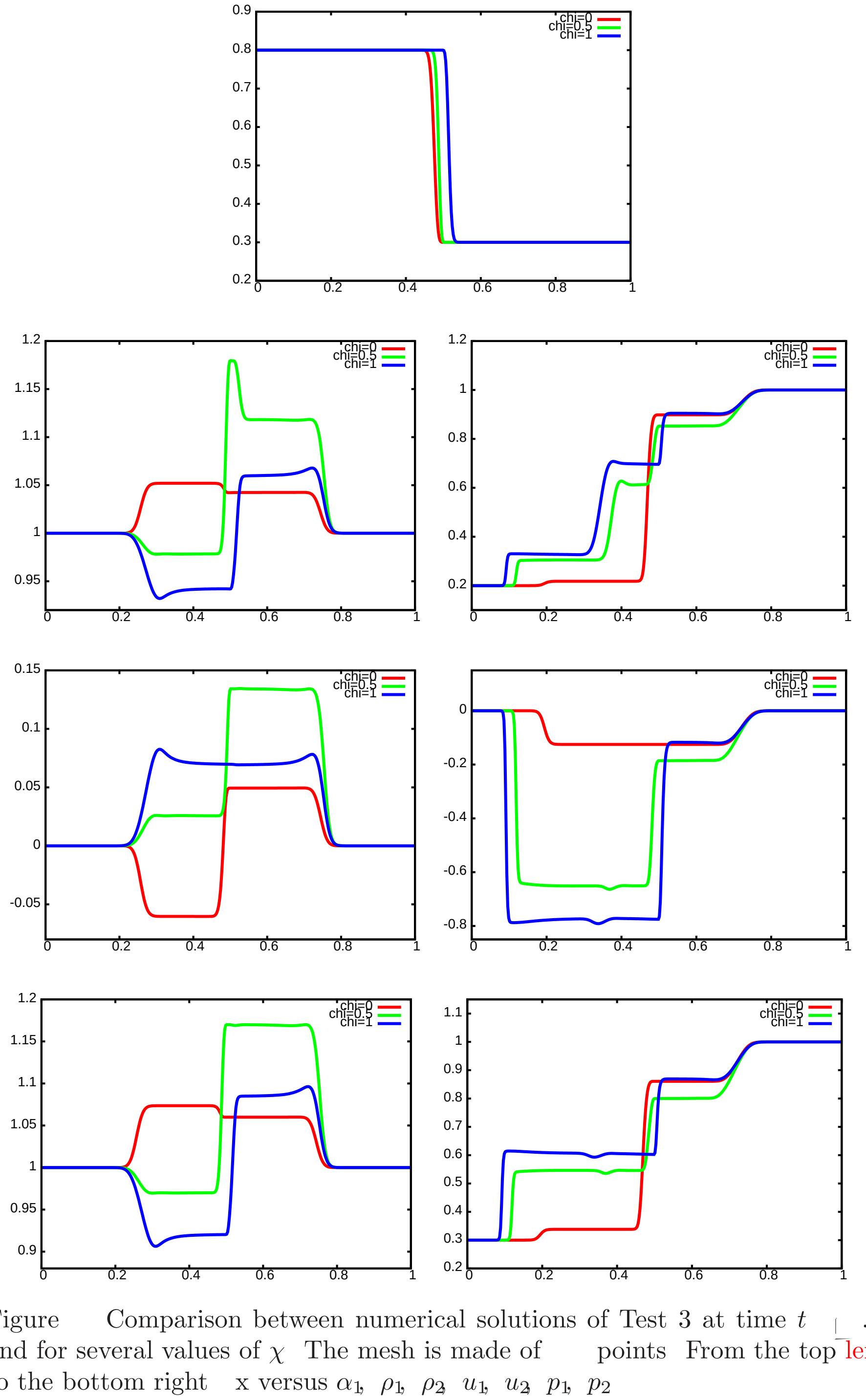

Figure Comparison between numerical solutions of Test 3 at time t . and for several values of χ The mesh is made of points From the top left to the bottom right x versus α_1, ρ_1, ρ_2, u_1, u_2, p_1, p_2

sources and the corresponding asymptotic properties The results given by the method are very satisfying but the capabilities of the scheme would be further emphasized by convergence studies with respect to the mesh size and by an extension of the method on D configurations

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