A finite volume approximation for second order elliptic problems with a full matrix on quadrilateral grids: derivation of the scheme and a theoretical analysis

A. Njifenjou[†]

[†] *The University of Yaounde I*
Ecole Nationale Supérieure Polytechnique
Department of Computer and Mathematical Engineering
P.O. Box: 8390 Yaounde - Cameroon

njifa2000@yahoo.fr

I. M. Nguena[⋆]

[⋆] *The University of Yaounde I*
Ecole Nationale Supérieure Polytechnique
Department of Computer and Mathematical Engineering
P.O. Box: 8390 Yaounde - Cameroon

imoukouop@yahoo.com

Abstract

We present in this paper a finite-volume based flexible Multi-Point Flux Approximation method (MPFA method, in short) displaying strong capabilities to handle flow problems in non-homogeneous anisotropic media. When the diffusion coefficient governing the flow is a full matrix with constant components, the discrete system is symmetric positive definite even if this matrix is only positive definite. In addition, if the diffusion coefficient is diagonal, the discrete system is reduced to two independent discrete systems corresponding to well-known cell-centered and vertex-centered finite volume methods.

A stability result and error estimates are given in L^2- and $L^\infty-norm$ and in a discrete energy norm as well. These results have been confirmed by numerical experiments. In this connection, a comparison of our MPFA method with the MPFA O-method has been performed.

Key words : diffusion problems, non-homogeneous anisotropic media, finite volume, multi-point flux approximation method, stability, error estimates.

1 Introduction

The diffusion phenomenon in heterogeneous anisotropic media (for instance, multiphase flow in geologically complex reservoirs) is one of the most important issues related to environmental problems. A better understanding of this phenomenon via numerical simulation could be achieved if challenging numerical schemes are put into codes. In this connection, several numerical methods are proposed by many authors. Among these methods, let us mention the most important one: (i) The Galerkin finite element that unfortunately displays limitations about the local conservativity (i.e. conservativity at grid block level) which is an essential condition in fluid flow simulation. (ii) The mixed and mixed hybrid finite element are two avatars of a same approach which is based upon a pressure-velocity formulation. This formulation meets the local conservativity and leads to satisfactory results. However some drawbacks concerning this formulation have been pointed out: for instance, violation of the discrete maximum principle near the well for flow within aquifers (see [H 02]). (iii) The finite volume, thanks to its great flexibility towards complex applications and its local conservativity, is considered today as the most comfortable numerical way for addressing fluid dynamics problems. Let us mention some recent significant contributions for finite volume solution of diffusion in anisotropic non-homogeneous media: (i) The Multi-Point Flux Approximation method (MPFA method): see for instance [ABBM 96a], [ABBM 98], [ABBM 96b], [MIS 03], [LDLC 98],[EK 05]; (ii) An extended K-orthogonal grid method has been developed for addressing diffusion problems with full matrix coefficients: see for instance [EGH 02].

This paper presents a new MPFA method involving the following innovations:

(i) Addressing an anisotropic diffusion problem by the MPFA O-method leads to an approximate pressure that is discontinuous on grid blocks boundaries. The variants of MPFA method proposed in this paper are based upon continuity in flux and pressure. The prize to pay is that one has to solve a discrete system involving discrete unknowns at the centers and vertices of the primary grid blocks.

(ii) The effective diffusion tensor governing large scale flow equations is not necessary symmetric (see [R 05] for instance). This work presents variants of MPFA that apply for flow equations involving spatially varying diffusion tensors (positive definite), not necessarily symmetric.

For presenting our finite volume formulations, let us consider the 2D diffusion problem consisting in finding a function that satisfies the following partial differential equation associated with a Dirichlet boundary condition:

$$-div(D\ grad\ U) = f \qquad \text{in} \qquad \Omega \tag{1.1}$$

$$U = 0 \quad \text{on} \qquad \Gamma \tag{1.2}$$

where f is a given function (commonly called source/sink term), Ω is a given open parallelogram domain and Γ denotes its boundary. $D = D(x)$, with $x = (x_1, x_2) \in \Omega$, is a full matrix describing the spatial variation of the diffusion coefficient which satisfies the uniform ellipticity i.e.

$$\exists \gamma \in \mathbb{R}_+^* \ \ such \ \ that\ \forall \varepsilon \in \mathbb{R}^2\ \varepsilon^T D(x)\varepsilon \geq \gamma\, |\varepsilon|^2 \quad a.e.\ \ in\ \ \Omega \tag{1.3}$$

where $| \,.\, |$ denotes the euclidian norm in $\mathbb{R}^2$, $D_{ij}(.)$ are the components of D and $L^\infty(\Omega)$ functions.

This paper is organized as follows. In the second section we present the spatial discretization and some important terminologies. The third section deals with the presentation of new variants of MPFA method for solving problem (1.1) – (1.2), while the fourth section investigates their theoretical properties in the case of an anisotropic homogeneous medium. In the fifth section, some numerical simulations are performed for homogeneous and non-homogeneous media. The sixth section is devoted to conclusions and perspectives of the work.

2 Spatial discretization

One starts with a grid consisting of a collection of polygonal cells perfectly defined by their corner points (*see* $Fig.$1 below showing an example of quadrilateral grid). The isobarycenter of a mesh cell is called a cell center or center point. The value of the numerical solution is computed at the corner and center points. We denote $x^{(i)}$ either a center or a corner point, and u_i the corresponding value of the numerical solution.

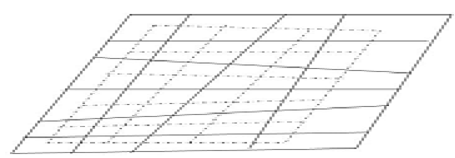


Figure 1: An example of quadrilateral grid (full lines) an its dual (dotted lines).

Let us introduce the notions of cluster and interaction region because of their key role in the discretization process. A collection of mesh cells with one common corner point is called a cluster. The number of cells in a cluster defines the degree of the corresponding corner point. Associated with each cluster is an interaction region which is defined as follows: On each cell edge, choose an arbitrary point $e^{(i)}$ and draw straight lines from this point to the center points in the two neighboring cells. Inside the cluster, the straight lines will define a polygon. That polygon defines an interaction region. Examples of clusters with interactions regions are presented in the following figure.

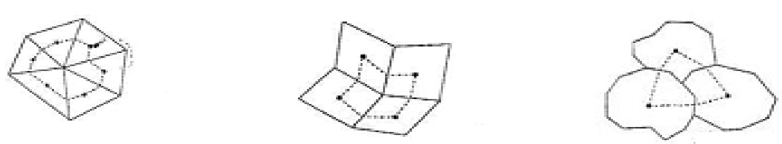


Figure 2: Miscellaneous examples of clusters (full lines) with their associated interaction regions (dotted lines).

The set of interaction regions defines a secondary mesh or dual mesh. It is then clear that every corner point is in fact a center point for the dual mesh. The initial mesh will be some times called the primary mesh and the dual mesh will be termed secondary mesh. Any discrete equation is associated with a cell point or a corner point, and is derived from the mass balance equation in a suitable control volume. More precisely, expressing the mass balance in each cell of the primary and secondary grids leads to the discrete problem. An original way to perform that mass balance is to focus on an interaction region in order to compute the contribution of each half edge to the mesh cells of the associated cluster (*see Fig* 3 below).

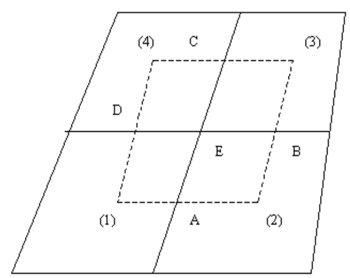


Figure 3: A cluster with its interaction region for computing the flux across [EA],[EB],[EC] and [ED] in a quadrilateral grid.

3 Finite volume formulation

3.1 Some new multi-point flux approximation methods

For sake of clarity, we present two novel multi-point flux approximation methods based upon a quadrilateral grid for solving the Dirichlet problem (1.1) – (1.2). We start with the mass balance equation in every cell C from the primary or secondary mesh:

$$\int_C -\, div\,[D\ grad\ u]\ \ dx = \int_C f(x)\ dx$$

Applying the Ostrograski theorem to the left hand side of this equality leads to what follows:

$$\int_{\Gamma_C} -\ \ [D\ grad\ U]\cdot n\ ds = \int_C f(x)\ dx$$

where Γ_C is the boundary of C, and where $\cdot$ denotes the standard scalar product in $\mathbb{R}^2$. The first integral term in this equality satisfies:

$$\int_{\Gamma_C} -\ \ [D\ grad\ U]\cdot n\ ds = -\sum_{[EA]}\int_{[EA]}[D\ grad\ U]\cdot n_{EA}\ ds$$

leading to the following approximation relation :

$$-\sum_{[EA]} \int_{[EA]} [D\ grad\ U] \cdot n_{EA}\ ds \approx -\sum_{[EA]} EA\left[(D\ grad\ U \cdot n_{EA})(x^A)\right] \tag{3.1}$$

where $[EA]$ is any half edge of C and where n_{EA} is the unit normal vector to $[EA]$ exterior to C, $x^A = (x_1^A, x_2^A)$ are the spatial coordinates of the point A. We perform the flux approximation in two manners: (i) demanding that the flux continuity be satisfied on each half edge; (ii) demanding that the flux continuity be satisfied on each entire edge. In a grid block, the degrees of freedom of the discrete pressure U are cell center and corners values. In what follows we present in details our strategy.

3.1.1 Derivation of the flux approximation imposing continuity per half edge

In what follows we are going to focus on the computation of the approximate flux across the halves edges, under the continuity condition.

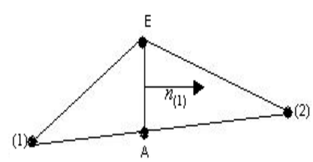


Figure 4: Flux molecule for formulating the flux continuity over the half edge $[EA]$

For performing the flux across the half edge $[EA]$ (see Fig. 4 above), one should calculate $\frac{\partial U}{\partial x_1}$ and $\frac{\partial U}{\partial x_2}$ at the point A (whose coordinates are $x^A = (x_1^A, x_2^A)$) in view to apply the approximation (3.1). The following change is made on the space variables:

$$\begin{cases} x_1 = x_1^A + (x_1^{(1)} - x_1^A)X_1 + (x_1^E - x_1^A)X_2 \\ \\ x_2 = x_2^A + (x_2^{(1)} - x_2^A)X_1 + (x_2^E - x_2^A)X_2 \end{cases}$$

Denoting $\det_{(1)}$ the determinant of the preceding system, it is easily seen that:

$$det_{(1)} = (x_1^{(1)} - x_1^A)(x_2^E - x_2^A) - (x_1^E - x_1^A)(x_2^{(1)} - x_2^A)$$

$$\frac{\partial X_1}{\partial x_1} = \frac{(x_2^E - x_2^A)}{det_{(1)}},\ \frac{\partial X_1}{\partial x_2} = \frac{(x_1^A - x_1^E)}{det_{(1)}},\ \frac{\partial X_2}{\partial x_1} = \frac{(x_2^A - x_2^{(1)})}{det_{(1)}},\ \frac{\partial X_2}{\partial x_2} = \frac{(x_1^{(1)} - x_1^A)}{det_{(1)}}$$

Accounting with the change of spatial variables and remembering that A is a point from the triangle $AE(1)$, the partial derivatives of U at x^A are given by:

$$\left(\frac{\partial U}{\partial x_1}\right)_{(1)}(x^A) = \frac{\partial U}{\partial X_1}\frac{\partial X_1}{\partial x_1} + \frac{\partial U}{\partial X_2}\frac{\partial X_2}{\partial x_1},\quad \left(\frac{\partial U}{\partial x_2}\right)_{(1)}(x^A) = \frac{\partial U}{\partial X_1}\frac{\partial X_1}{\partial x_2} + \frac{\partial U}{\partial X_2}\frac{\partial X_2}{\partial x_2}$$

Then these partial derivatives are approximated as follows:

$$\frac{\partial U}{\partial X_1}\frac{\partial X_1}{\partial x_1}+\frac{\partial U}{\partial X_2}\frac{\partial X_2}{\partial x_1}\approx\left(U_{(1)}-U_A\right)\frac{(x_2^E-x_2^A)}{det_{(1)}}+(U_E-U_A)\frac{(x_2^A-x_2^{(1)})}{det_{(1)}}\tag{3.2}$$

$$\frac{\partial U}{\partial X_1}\frac{\partial X_1}{\partial x_2}+\frac{\partial U}{\partial X_2}\frac{\partial X_2}{\partial x_2}\approx\left(U_{(1)}-U_A\right)\frac{(x_1^A-x_1^E)}{det_{(1)}}+(U_E-U_A)\frac{(x_1^{(1)}-x_1^A)}{det_{(1)}}\tag{3.3}$$

On the other hand, the normal vector $n_{(1)}$ is defined as:

$$n_{(1)}=\varepsilon\left(\frac{x_2^E-x_2^A}{AE},\frac{x_1^A-x_1^E}{AE}\right)^t$$

where $\varepsilon=\pm1$ is chosen in a way that $n_{(1)}$ is steered towards the outside of the triangle $AE(1)$ (see $Figure$ 4 above).
We deduce that :

$$[D\ grad\ U]\cdot n_{(1)}=\varepsilon\left[\frac{x_2^E-x_2^A}{AE}\ \ \frac{x_1^A-x_1^E}{AE}\right]\left(\begin{bmatrix}D_{11}&D_{12}\\D_{21}&D_{22}\end{bmatrix}\begin{bmatrix}\frac{\partial U}{\partial x_1}\\\frac{\partial U}{\partial x_2}\end{bmatrix}\right)$$

Setting

$$\alpha_{(1)}^E=\varepsilon\left(D_{11}\frac{x_2^E-x_2^A}{AE}+D_{21}\frac{x_1^A-x_1^E}{AE}\right),\ \beta_{(1)}^E=\varepsilon\left(D_{12}\frac{x_2^E-x_2^A}{AE}+D_{22}\frac{x_1^A-x_1^E}{AE}\right)$$

it follows that

$$(D\ grad\ U)\cdot n_{(1)}=\alpha_{(1)}^E\frac{\partial U}{\partial x_1}\left(x^A\right)+\ \beta_{(1)}^E\frac{\partial U}{\partial x_2}\left(x^A\right)$$

Recall that approximations of partial derivatives of U at x^A are given by relations (3.2) – (3.3), when considering that A is a point of the triangle $EA(1)$. Similarly, considering that A is also a point from the triangle $EA(2)$, we have

$$\begin{aligned}\left(\frac{\partial U}{\partial x_1}\right)_{(2)}\left(x^A\right)&=\left(U_{(2)}-U_A\right)\frac{(x_2^E-x_2^A)}{det_{(2)}}+(U_E-U_A)\frac{(x_2^A-x_2^{(2)})}{det_{(2)}}\\\left(\frac{\partial U}{\partial x_2}\right)_{(2)}\left(x^A\right)&=\left(U_{(2)}-U_A\right)\frac{(x_1^A-x_1^E)}{det_{(2)}}+(U_E-U_A)\frac{(x_1^{(2)}-x_1^A)}{det_{(2)}}\end{aligned}$$

where we have set

$$det_{(2)}=(x_1^{(2)}-x_1^A)(x_2^E-x_2^A)-(x_1^E-x_1^A)(x_2^{(2)}-x_2^A)$$

Therefore, the flux continuity equation on $[EA]$ reads:

$$\alpha_{(1)}^E\left(\frac{\partial U}{\partial x_1}\right)_{(1)}+\ \beta_{(1)}^E\left(\frac{\partial U}{\partial x_2}\right)_{(1)}+\alpha_{(2)}^E\left(\frac{\partial U}{\partial x_1}\right)_{(2)}+\ \beta_{(2)}^E\left(\frac{\partial U}{\partial x_2}\right)_{(2)}=0\tag{3.4}$$

Setting for $i = 1, 2$:

$$C_{11}^{(i)E} = \frac{x_2^E - x_2^A}{det_{(i)}}, \quad C_{12}^{(i)E} = \frac{x_2^A - x_2^{(i)}}{det_{(i)}}, \quad C_{21}^{(i)E} = \frac{x_1^A - x_1^E}{det_{(i)}}, \quad C_{22}^{(i)E} = \frac{x_1^{(i)} - x_1^A}{det_{(i)}}$$

and for $i,\ j = 1, 2$:

$$S_{1j}^{(i)E} = \alpha_{(i)}^E C_{1j}^{(i)E} + \ \beta_{(i)}^E C_{2j}^{(i)E}$$

one deduces from the discrete version of the flux continuity equation (3.4) that:

$$U_A = \frac{S_{11}^{(1)E} U_{(1)} + (S_{12}^{(1)E} + S_{12}^{(2)E}) U_E + S_{11}^{(2)E} U_{(2)}}{S_{11}^{(1)E} + S_{12}^{(1)E} + S_{12}^{(2)E} + S_{11}^{(2)E}} \equiv \gamma_{(1)}^E U_{(1)} + \gamma_E^E U_E + \gamma_{(2)}^E U_{(2)}$$

Setting also:

$$\begin{aligned} \Phi_{(1)} \quad &= \quad U_{(1)}(S_{11}^{(1)E} - \gamma_{(1)}^E (S_{11}^{(1)E} + S_{12}^{(1)E})) + U_E((S_{12}^{(1)E} - \gamma_E^E (S_{11}^{(1)E} + S_{12}^{(1)E})) \\ &\quad -\gamma_{(2)}^E U_{(2)} (S_{11}^{(1)E} + S_{12}^{(1)E}) \end{aligned}$$

the flux across $[EA]$ may be written finally as follows:

$$\Phi_{[EA]}^{(1)} \ = \ - \ AE \ \Phi_{(1)} \tag{3.5}$$

One should note that the methodology presented above is completely general. This means it applies to all the interior halves edges of cells from the primary and secondary grids. On the other hand, for a boundary half-edge $[EA]$, there is no need to write the continuity equation since U_A is given by the Dirichlet condition. Thus, the global discrete system may be easily derived following what precedes.

Remark 1 In the case of a Dirichlet-Neumann boundary problem, one should deal with the case where a half edge $[EA]$ is included in Neumann's boundary. The flux across $[EA]$ is actually given by the Neumann condition $\cdot$

3.1.2 Derivation of the flux approximation imposing continuity per edge

In the previous subsection, the mass balance is performed under the constraint of the flux continuity on every half edge. Unfortunately, it may happen when dealing with the primary mesh, that the expression $S_{11}^{(1)E} + S_{12}^{(1)E} + S_{12}^{(2)E} + S_{11}^{(2)E}$ be null, and therefore it is not possible to compute $\Phi_{[EA]}^{(1)}$. To overcome this difficulty, one may write the flux continuity equation on the entire edge $[EF]$ (see $Figure$ 5 below).

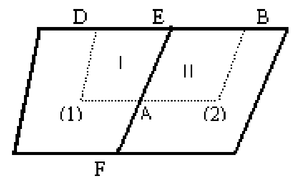


Figure 5: Flux molecule for formulating the flux continuity over the edge $[EF]$

For this purpose, let us set:

$$\Phi_{(1)}^{[EF]} = AE\left[\alpha_{(1)}^{E}\left(\frac{\partial U}{\partial x_1}\right)_{(1)} + \beta_{(1)}^{E}\left(\frac{\partial U}{\partial x_2}\right)_{(1)}\right] \qquad +AF\left[\alpha_{(1)}^{F}\left(\frac{\partial U}{\partial x_1}\right)_{(1)} + \beta_{(1)}^{F}\left(\frac{\partial U}{\partial x_2}\right)_{(1)}\right]$$

$$\Phi_{(2)}^{[EF]} = AE\left[\alpha_{(2)}^{E}\left(\frac{\partial U}{\partial x_1}\right)_{(2)} + \beta_{(2)}^{E}\left(\frac{\partial U}{\partial x_2}\right)_{(2)}\right] \qquad +AF\left[\alpha_{(2)}^{F}\left(\frac{\partial U}{\partial x_1}\right)_{(2)} + \beta_{(2)}^{F}\left(\frac{\partial U}{\partial x_2}\right)_{(2)}\right]$$

Therefore the flux continuity equation over $[EF]$ reads :

$$\Phi_{(1)}^{[EF]} + \Phi_{(2)}^{[EF]} = 0 \tag{3.6}$$

The last equation permits to express the numerical potential U_A in terms of $U_{(1)}$, $U_{(2)}$, U_E and U_F. To solve this equation, one may consider two cases: the case where the elements I and II (see $Figure$ 5 above) are from the primary mesh, and the case where these elements are from the secondary mesh.

First case: the elements I and II are from the primary grid. Solving the flux continuity equation (3.6), where U_A is the unknown, leads to the following expression of the flux $\Phi_{(1)}^{[EF]}$:

$$\Phi_{(1)}^{[EF]} = -EF\left[c_1(U_{(1)} - U_{(2)}) + rc_2(U_E - U_F)\right] \tag{3.7}$$

where we have set

$$c_1 = \frac{S_{11}^{(1)E} S_{11}^{(2)E}}{S_{11}^{(1)E} + S_{11}^{(2)E}}, \qquad c_2 = \frac{S_{12}^{(1)E} S_{11}^{(2)E} - S_{11}^{(1)E} S_{12}^{(2)E}}{S_{11}^{(1)E} + S_{11}^{(2)E}} \qquad \text{and} \qquad r = \frac{EA}{EF} \tag{3.8}$$

Second case: the elements I and II are from the secondary grid. In this case the flux across $[EF]$ reads as follows:

$$\Phi_{(1)}^{[EF]} = c_1 U_{(1)} + c_2 U_{(2)} + AE\ S_{12}^{(1)E}\ U_E + AF\ S_{12}^{(1)F}\ U_F \tag{3.9}$$

where we have set:

$$c_1 = -\frac{1}{2}\left(AE\ S_{11}^{(1)E} + AF\ S_{11}^{(1)F} + AE\ S_{12}^{(1)E} + AF\ S_{12}^{(1)F}\right) \tag{3.10}$$

$$c_2 = \frac{1}{2}\left(AE\ S_{11}^{(1)E}\ + AF\ S_{11}^{(1)F}\ - AE\ S_{12}^{(1)E}\ - AF\ S_{12}^{(1)F}\right) \tag{3.11}$$

Remark 2 Due to the discontinuity of the permeability, the expression $S_{11}^{(1)E} + S_{12}^{(1)E} + S_{12}^{(2)E} + S_{11}^{(2)E}$ may be equal to zero for half edge computation of flux in the primary grid, as one can see in the following example. Let Ω be the square $]0,1[^2$ made of two structures characterized by the permeabilities :

$$D = D(x_1, x_2) = \begin{bmatrix} 1 & 2 \\ 2 & 10 \end{bmatrix} \quad \text{if } x_1 \leq 0.5$$

and

$$D = D(x_1, x_2) = \begin{bmatrix} 1 & 0 \\ 0 & 20 \end{bmatrix} \qquad \text{otherwise.}$$

One easily checks that in a primary uniform rectangular grid, $S_{11}^{(1)E} + S_{12}^{(1)E} + S_{12}^{(2)E} + S_{11}^{(2)E} = 0$ for every half edge included in the line $x_1 = 0.5$. In another hand, this kind of singularity should not occur for half edge or entire edge computation of flux in the dual grid. Indeed, when dealing with the dual grid, the expression $S_{11}^{(1)E} + S_{12}^{(1)E} + S_{12}^{(2)E} + S_{11}^{(2)E}$ is reduced to $S_{11}^{(1)E} + S_{11}^{(2)E}$which is never null •

3.1.3 Combining the two approaches

One can prove that $S_{11}^{(1)E} + S_{12}^{(1)E} + S_{12}^{(2)E} + S_{11}^{(2)E}$ could be null in the scheme (3.5) only when dealing with cells from the primary grid. It is therefore suitable to use this scheme for cells of the secondary grid. The schemes $(3.7)-(3.11)$ apply for cells from the primary grid and those from the secondary grid as well.

3.2 Definition of an approximate solution in terms of function

Recall that the proposed variants of MPFA method lead to solving a square system of equations involving all the discrete unknowns U_m representing the approximate pressure at cell centers and cell corners (with respect to the primary grid). Once these unknowns are computed, one utilizes them as interpolation data in the construction of an approximate solution denoted U_h defined as a piecewise linear function (more details are given below).

The computed quantities U_m actually correspond to the values of U_h at cell centers and cell corners. Thus these quantities satisfy the relation

$$U_m = U_h(x^{(m)}), \text{where } x^{(m)} \text{ is a node i.e. a cell center or a corner point.}$$

Let us say more about the definition of U_h. Consider a grid cell of the primary grid. To define U_h at every point of that grid cell, we divide it into four triangular elements constructed by joining the cell center to the four cell corners (see $Figure$ 6 below):

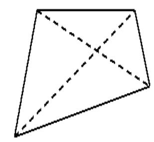


Figure 6: A grid block divided into four triangles for a piecewise linear approximation of the solution

DEFINITION 3.1 Let $x^{(i)}$, $x^{(j)}$ and $x^{(k)}$ denote the vertices of a triangular element T obtained from the division of a primary grid cell. The approximate solution U_h of the diffusion problem (1.1) – (1.2) is defined in T as follows:

$$U_h(x) = \alpha \cdot (x - x^{(i)}) + U_i \tag{3.12}$$

where $x = (x_1, x_2)^t$, $\alpha = (\alpha_1, \alpha_2)^t$, $x^{(i)} = (x_1^{(i)}, x_2^{(i)})^t$ and $U_i = U_h(x^{(i)})$, with $(.,.)^t$ denoting the transposition operator. The components of the vector α are easily calculated thanks to the fact that $U_j = U_h(x^{(j)})$ and $U_k = U_h(x^{(k)})$ are given (from the solution of the finite volume discrete system)•

PROPOSITION 3.2 *The approximate solution U_h is a continuous function in $\overline{\Omega}$ (closure of Ω). Moreover U_h is in the space $H_0^1(\Omega)$*•

Before carrying out the proof of this Proposition, let us recall that for an open subset D of $\mathbb{R}^2$, the spaces $H^1(D)$ and $H_0^1(D)$ are defined as follows:

$$H^1(D) = \left\{ v \in L^2(D)\,;\ \frac{\partial v}{\partial x_i} \in L^2(D) \quad for \quad i = 1,2 \right\} \tag{3.13}$$

$$H_0^1(D) = \left\{ v \in H^1(D); \quad v = 0 \quad \text{on} \quad \Gamma \right\} \tag{3.14}$$

where $L^2(D)$ is the space (of classes) of functions v such that $\int_D v^2 dx$ converges, and where $\frac{\partial v}{\partial x_i}$ denotes partial derivatives in a distributional sense (see for instance [B83] for more details). The mapping

$$v \longmapsto \left[\int_\Omega |grad\ v|^2\, dx \right]^{\frac{1}{2}} \tag{3.15}$$

defines the well-known $H_0^1(\Omega) - norm$.

Proof. One easily checks that U_h is continuous on grid blocks boundaries. This follows from the fact that U_h is linear in each triangular element T (inside grid blocks) and is continuous at the corner points (of the primary grid). So U_h is continuous over $\overline{\Omega}$. Since the restriction of U_h in each triangular element T is in $H^1(T)$ and U_h takes zero value on the boundary Γ of Ω, then U_h is in $H_0^1(\Omega)$ ■

4 Analysis of the proposed Finite Volume Method

We are going to focus on the case of diffusion problems governed by uniform full diffusion tensors (not necessarily symmetric) i.e. tensors without dependency on spatial variables. This particular case is of great importance for some applied problems arising from homogenization theory. Indeed when the domain Ω is made of a periodic microstructure, the diffusion coefficient is a periodic function whose reference period is the unit cell of Ω. The classical theory of homogenization based upon the multi-scale method shows that Ω behaves like an homogeneous medium characterized by an effective full diffusion tensor D (see [SP 80], [N 94], [BLP 78] for instance). In the sequel we denote D_{11}, D_{12}, D_{21} and $D_{22}$the components of that effective diffusion tensor. Some authors have extended the homogenization theory to non-periodic media (see for instance [R 05], [D 05]).

4.1 Definition of the discrete system

We present in this sub-section the matrix form of our finite volume formulation for $(1.1) - (1.2)$. For sake of clarity of the presentation we suppose that the spatial domain Ω is $]0,1[\times]0,1[$. We assume that Ω is covered with a square grid whose size is $h = \frac{1}{N}$, where N is a given positive integer. On the other hand, we denote K_{ij} the grid block defined by :

$K_{ij} = \left[x_1^{i-\frac{1}{2}}, x_1^{i+\frac{1}{2}}\right] \times \left[x_2^{j-\frac{1}{2}}, x_2^{j+\frac{1}{2}}\right]$ where $x_1^{i+\frac{1}{2}} = x_1^{i-\frac{1}{2}} + h, \quad x_2^{j+\frac{1}{2}} = x_2^{j-\frac{1}{2}} + h,\ for\ i,j = 1,\dots,N$ with $x_1^{\frac{1}{2}} = x_2^{\frac{1}{2}} = 0$

From the boundary value problems theory (see for instance [B 83]) it is clear that under the following additional assumption :

$$f \in C^m\left(\overline{\Omega}\right),\ \text{where } m \in \mathbb{N} \text{ is given,} \tag{4.1}$$

the problem $(1.1) - (1.2)$ possesses a unique solution φ in $C^{m+2}\left(\overline{\Omega}\right)$. In what follows, we suppose that f is a continuous function in $\overline{\Omega}$. Therefore a unique solution φ for $(1.1) - (1.2)$ exists in $C^2\left(\overline{\Omega}\right)$. Utilizing a finite volume formulation of the problem $(1.1) - (1.2)$ we are going to derive a discrete system involving $\{u_{i,j}\}_{1\le i,j\le N}$ and $\left\{u_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\le i,j\le N-1}$ as discrete unknowns expected to be reasonable approximations of $\{\varphi_{i,j}\}_{1\le i,j\le N}$ and $\left\{\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\le i,j\le N-1}$ respectively, where $\varphi_{i,j} = \varphi\left(x_1^i, x_2^j\right)$ with $x_1^i = \frac{x_1^{i-\frac{1}{2}}+x_1^{i+\frac{1}{2}}}{2}$, $x_2^i = \frac{x_2^{j-\frac{1}{2}}+x_2^{j+\frac{1}{2}}}{2}$ and $\varphi_{i+\frac{1}{2},j+\frac{1}{2}} = \varphi\left(x_1^{i+\frac{1}{2}}, x_2^{j+\frac{1}{2}}\right)$.

Let us now integrate (1.1) in the grid block K_{ij}, usually called a control volume, centered at the point $\left(x_1^i, x_2^j\right)$. Applying Ostrogradski theorem and a quadrature formula for approximating the flux on the boundary of K_{ij} leads to the relation :

$$\begin{aligned}
&D_{21}\left[\varphi_{i-\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\right]+D_{22}\left[\varphi_{i,j}-\varphi_{i,j+1}\right]+D_{21}\left[\varphi_{i+\frac{1}{2},j-\frac{1}{2}}-\varphi_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\
&+D_{22}\left[\varphi_{i,j}-\varphi_{i,j-1}\right]+D_{12}\left[\varphi_{i+\frac{1}{2},j-\frac{1}{2}}-\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\right]+D_{11}\left[\varphi_{i,j}-\varphi_{i+1,j}\right]\\
&+D_{12}\left[\varphi_{i-\frac{1}{2},j+\frac{1}{2}}-\varphi_{i-\frac{1}{2},j-\frac{1}{2}}\right]+D_{11}\left[\varphi_{i,j}-\varphi_{i-1,j}\right]\\
&\approx \int_{K_{ij}} f(x)dx \qquad\qquad \forall \quad 1\leq\ i,\ j\leq\ N
\end{aligned}\tag{4.2}$$

From the boundary conditions (1.2) we have the following relations

$$\varphi_{i+\frac{1}{2},\frac{1}{2}}=\varphi_{i+\frac{1}{2},N+\frac{1}{2}}=\varphi_{\frac{1}{2},j+\frac{1}{2}}=\varphi_{N+\frac{1}{2},j+\frac{1}{2}}=0 \qquad \forall \quad 0\leq\ i,\ j\leq\ N \tag{4.3}$$

On the other hand, we adopt the natural convention that :

$$\varphi_{i,0}=\varphi_{0,j}=\varphi_{i,N+1}=\varphi_{N+1,j}=0 \qquad \forall \quad 1\leq\ i,\ j\leq\ N \tag{4.4}$$

Since $\varphi=0$ on Γ (see equation (1.2)), one naturally extends φ by zero in $\mathbb{R}^2\setminus\,\Omega$.

The discrete system $(4.2)-(4.4)$ is not closed since the number of unknowns is greater than the number of equations. Indeed there are $[N^2{+}(N{-}1)^2{+}4N{+}4(N{-}1)]$ unknowns and only $[N^2{+}4N{+}4(N{-}1)]$ equations. Therefore we should add $(N{-}1)^2$ equations to that system.

For this purpose it is natural to integrate the balance equation (1.1) in the finite volume $K_{i+\frac{1}{2},j+\frac{1}{2}}=\left[x_1^i,x_1^{i+1}\right]\times\left[x_2^j,x_2^{j+1}\right]$. Applying one time more Ostrogradski theorem and a quadrature formula for approximating the flux on the boundary of $K_{i+\frac{1}{2},j+\frac{1}{2}}$ leads to the following relation:

$$\begin{aligned}
&D_{21}\left[\varphi_{i,j+1}-\varphi_{i+1,j+1}\right]+D_{22}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j+\frac{3}{2}}\right]\\
&+D_{21}\left[\varphi_{i+1,j}-\varphi_{i,j}\right]+D_{22}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j-\frac{1}{2}}\right]\\
&+D_{12}\left[\varphi_{i+1,j}-\varphi_{i+1,j+1}\right]+D_{11}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{3}{2},j+\frac{1}{2}}\right]\\
&+D_{12}\left[\varphi_{i,j+1}-\varphi_{i,j}\right]+D_{11}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i-\frac{1}{2},j+\frac{1}{2}}\right]\\
&\qquad\qquad\approx \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} f(x)dx \quad \forall \quad 1\leq\ i,\ j\leq\ N\ -1
\end{aligned}\tag{4.5}$$

From the closed discrete system (4.2)-(4.5) we derive the discrete problem consisting in finding $\{u_{i,j}\}_{1\le i,j\le N}$ and $\left\{u_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\le i,j\le N-1}$ real numbers such that:

$$
\begin{aligned}
&D_{21}\left[u_{i-\frac{1}{2},j+\frac{1}{2}}-u_{i+\frac{1}{2},j+\frac{1}{2}}\right]+D_{22}\left[u_{i,j}-u_{i,j+1}\right]+D_{21}\left[u_{i+\frac{1}{2},j-\frac{1}{2}}-u_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\
&+D_{22}\left[u_{i,j}-u_{i,j-1}\right]+D_{12}\left[u_{i+\frac{1}{2},j-\frac{1}{2}}-u_{i+\frac{1}{2},j+\frac{1}{2}}\right]+D_{11}\left[u_{i,j}-u_{i+1,j}\right]\\
&+D_{12}\left[u_{i-\frac{1}{2},j+\frac{1}{2}}-u_{i-\frac{1}{2},j-\frac{1}{2}}\right]+D_{11}\left[u_{i,j}-u_{i-1,j}\right]\\
&=\int_{K_{ij}} f(x)dx \qquad\qquad \forall\quad 1\le\ i,\ j\le\ N
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
&D_{21}\left[u_{i,j+1}-u_{i+1,j+1}\right]+\ D_{22}\left[u_{i+\frac{1}{2},j+\frac{1}{2}}-u_{i+\frac{1}{2},j+\frac{3}{2}}\right]\\
&+D_{21}\left[u_{i+1,j}-u_{i,j}\right]+D_{22}\left[u_{i+\frac{1}{2},j+\frac{1}{2}}-u_{i+\frac{1}{2},j-\frac{1}{2}}\right]\\
&+D_{12}\left[u_{i+1,j}-u_{i+1,j+1}\right]+\ D_{11}\left[u_{i+\frac{1}{2},j+\frac{1}{2}}-u_{i+\frac{3}{2},j+\frac{1}{2}}\right]\\
&+D_{12}\left[u_{i,j+1}-u_{i,j}\right]+\ D_{11}\left[u_{i+\frac{1}{2},j+\frac{1}{2}}-u_{i-\frac{1}{2},j+\frac{1}{2}}\right]\\
&=\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} f(x)dx \qquad\qquad \forall\quad 1\le\ i,\ j\le\ N\ -1
\end{aligned}
\tag{4.7}
$$

where we have set

$$u_{i+\frac{1}{2},\frac{1}{2}}=u_{i+\frac{1}{2},N+\frac{1}{2}}=u_{\frac{1}{2},j+\frac{1}{2}}=u_{N+\frac{1}{2},j+\frac{1}{2}}=0 \qquad \forall\quad 0\le\ i,\ j\le\ N \tag{4.8}$$

and

$$u_{i,0}=u_{0,j}=u_{i,N+1}=u_{N+1,j}=0 \qquad \forall\quad 1\le\ i,\ j\le\ N \tag{4.9}$$

4.2 Existence and uniqueness for a solution of the discrete system

We are going to deal know with the existence and uniqueness for a solution of the discrete problem (4.6)-(4.7). Before giving the two main results of this sub-section, let us comment about this discrete problem. Its matrix form may be expressed as follows:

$$\begin{pmatrix} A & B\\ B^T & C\end{pmatrix}\begin{pmatrix} U_{cc}\\ U_{vc}\end{pmatrix}=\begin{pmatrix} F_{cc}\\ F_{vc}\end{pmatrix} \tag{4.10}$$

where we have set :

$$U_{cc}=\{u_{i,j}\}_{1\le i,j\le N} \qquad \text{and} \qquad U_{vc}=\left\{u_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\le i,j\le N-1} \tag{4.11}$$

and where:

F_{cc} is a sub-vector with N^2 components depending on the right hand side of (4.6) only as we account with (4.8) and (4.9).

F_{vc} is a sub-vector with $(N-1)^2$ components depending on the right hand side of (4.7) only as we account with (4.8) and (4.9).

A is a $N \times N$ symmetric positive definite matrix, associated to the classical cell-centered finite volume when D is diagonal i.e.$D_{12} = D_{21} = 0$.

C is a $(N-1) \times (N-1)$ symmetric positive definite matrix, associated to the classical vertex-centered finite volume when D is diagonal.

B is a $N \times (N-1)$ matrix and B^T is its transpose.

When the diffusion coefficient D is reduced to a diagonal matrix, the discrete problem (4.6)-(4.7) admits the following matrix form

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} U_{cc} \\ U_{vc} \end{pmatrix} = \begin{pmatrix} F_{cc} \\ F_{vc} \end{pmatrix} \tag{4.12}$$

When solving this system on a parallel computer, the CPU time is the same as when solving only one of the two following systems :

$$AU_{cc} = F_{cc} \quad \text{and} \qquad CU_{vc} = F_{vc}$$

Since A and C are both positive definite, the existence and the uniqueness for a solution of (4.12) are ensured. Therefore when D is diagonal, our formulation provides more information about the solution of the diffusion problem than the classical finite volume formulations, for the same CPU on a parallel computer. Let us give now the two main results of this subsection.

Proposition 4.1 The discrete problem consisting to find $\{u_{i,j}\}_{1 \le i,j \le N}$ and $\left\{u_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1 \le i,j \le N-1}$ such that the equations (4.6)-(4.7) are satisfied under the conditions (4.8) and (4.9), possesses a unique solution ·

Proposition 4.2 The matrix $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ associated to the discrete problem (4.6)-(4.7) is symmetric and positive definite ·

Since *Proposition* 4.2 implies *Proposition* 4.1, let us focus on the proof of the last proposition.

Proof. The symmetric structure of the matrix $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ is obvious from the comments between (4.11) and (4.12). Multiplying (4.6) by $u_{i,j}$ and (4.7) by $u_{i+\frac{1}{2},j+\frac{1}{2}}$ and summing leads to (here notations (4.11) are utilized) :

$$
\begin{aligned}
&[U_{cc}\ U_{vc}]\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}\begin{bmatrix} U_{cc} \\ U_{vc} \end{bmatrix} = \Big\{D_{22}\,(a-b)^2 + D_{21}(c-d)a + D_{21}(d-c)b \\
&\qquad + D_{11}\,(c-d)^2 + D_{12}(b-a)d + D_{12}(a-b)c\Big\} \\
&\qquad + \Big\{D_{22}\,(e-d)^2 + D_{21}(a-g)e + D_{21}(g-a)d \\
&\qquad + D_{11}\,(a-g)^2 + D_{12}(d-e)g + D_{12}(e-d)a\Big\} + \ ... = \\
&= \quad \Big\{D_{11}\,(c-d)^2 + D_{22}(a-b)^2 + D_{12}(c-d)(a-b) + D_{21}\,(c-d)\,(a-b)\Big\} \\
&+\Big\{D_{11}\,(a-g)^2 + D_{22}(e-d)^2 + D_{12}(a-g)(e-d) + D_{21}(a-g)(e-d)\Big\} + \ ... \\
&\qquad\qquad \text{(with a finite number of brackets)}
\end{aligned}
\tag{4.13}
$$

where we have set:

$$
\begin{aligned}
&a = u_{i,j} \quad , \quad b = u_{i,j+1} \quad , c = u_{i-\frac{1}{2},j+\frac{1}{2}} \quad , \\
&d = u_{i+\frac{1}{2},j+\frac{1}{2}} \quad , \quad e = u_{i+\frac{1}{2},j-\frac{1}{2}} \quad , \quad g = u_{i+1,j}
\end{aligned}
\tag{4.14}
$$

Thanks to the assumption (1.3) we deduce from what precedes that

$$
\begin{aligned}
&[U_{cc}\ U_{vc}]\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}\begin{bmatrix} U_{cc} \\ U_{vc} \end{bmatrix} \geq \gamma\left[(a-d)^2 + (a-b)^2\right] \\
&+\gamma\left[(a-g)^2 + (e-d)^2\right] + ... \\
&= \gamma \sum_{\substack{1\leq i\leq N \\ 0\leq j\leq N}} \left[(u_{i,j} - u_{i,j+1})^2 + (u_{i-\frac{1}{2},j+\frac{1}{2}} - u_{i+\frac{1}{2},j+\frac{1}{2}})^2\right] \\
&+\gamma \sum_{\substack{0\leq i\leq N \\ 1\leq j\leq N}} \left[(u_{i,j} - u_{i+1,j})^2 + (u_{i+\frac{1}{2},j-\frac{1}{2}} - u_{i+\frac{1}{2},j+\frac{1}{2}})^2\right] \ \geq 0
\end{aligned}
\tag{4.15}
$$

Thanks to the relations (4.8) and (4.9) the equality holds in (4.15) if and only if $U_{cc} = 0$ and $U_{vc} = 0$. Thus the positive definiteness of the matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ is proved. Therefore, it is clear that the discrete problem consisting in finding $\{u_{i,j}\}_{1\leq i,j\leq N}$ and $\left\{u_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\leq i,j\leq N-1}$ such that the equations (4.6)-(4.7) are satisfied possesses a unique solution. ■

4.3 Stability and error estimates

4.3.1 Preliminaries

We start by considering an other grid $\mathcal{L}$ (*see Figure* 7 below). The elements of $\mathcal{L}$ are made of lozenges L completely imbedded in $\overline{\Omega}$. We denote Γ_L the boundary of $L \in \mathcal{L}$ and $\mathbf{E}(\mathcal{L})$ the space of functions v defined almost everywhere in $\mathbb{R}^2$ such that

v is constant in $L \in \mathcal{L}$ and zero elsewhere. This space is obviously non-empty since there is the null function.

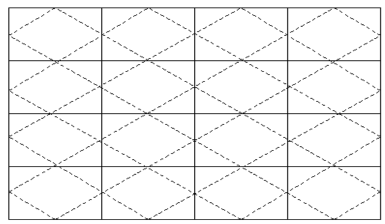


Figure 7: An example of grid $\mathcal{L}$ made of lozenges associated with a primary rectangular grid.

Let us endow $\mathbf{E}(\mathcal{L})$ with the following discrete energy norm. For all $v \in \mathbf{E}(\mathcal{L})$ we set:

$$\|v\|_{1,h} = \left[\sum_{s\in S} (\Delta_s v)\right]^{\frac{1}{2}} \tag{4.16}$$

where

$$(\Delta_s v) = \sum_{\substack{L,K \quad s,t \\ \Gamma_K \cap \Gamma_L = \{s\}}} |v_L - v_K|^2 \tag{4.17}$$

and where S is the set of vertices and K, L are taken in $\mathcal{L}$.

Note that a vertex $s \in S$ could belong to the boundary Γ of the domain Ω. In this case there exists a unique element L of $\mathcal{L}$ such that s belongs to the boundary Γ_L of L. It is therefore natural to define $(\Delta_s v)$ in this case by $(\Delta_s v) = |v_L|^2$. The norm defined by (4.16) could be viewed as a discrete version of the classical $H_0^1(\Omega)$ norm.

Let us introduce the space

$$C_0\left(\overline{\Omega}\right) = \{v : \overline{\Omega} \longrightarrow \mathbb{R} \text{ is continuous, and } v = 0 \text{ on } \Gamma\}$$

and the following operator :

$$\Pi_Q : \quad C_0\left(\overline{\Omega}\right) \longrightarrow E\left(\mathcal{L}\right)$$

$$v \mapsto \Pi_Q v$$

with:

$$[\Pi_Q v](x) = \begin{cases} v(x_{_L}), \text{ if } x \in \ Int(L) \\ \\ 0 \text{ if } \ x \in \ \mathbb{R}^2 \backslash \Omega \end{cases}$$

where $L \in \mathcal{L}$ and where $x_{_L} = (x_1^L, x_2^L)$ are the coordinates of the center of L.

Since the approximate solution U_h of the diffusion problem $(1.1) - (1.2)$ is in $C_0\left(\overline{\Omega}\right)$ (see subsection 3.2 for the definition of U_h), we have:

$$[\Pi_Q U_h](x) = \left\{ \begin{array}{ll} U_L, & \text{if } x \in Int(L) \\ 0 & \text{if } x \in \mathbb{R}^2 \backslash \Omega \end{array} \right\} \tag{4.18}$$

DEFINITION 4.3 Let v be a function of $\mathbf{E}(\mathcal{L})$. $v_{|\Omega}$ is a weak approximate solution for the diffusion problem $(1.1)-(1.2)$ if there exists an approximate solution V of $(1.1)-(1.2)$ in the sense of *Definition* 3.1 such that $v = \Pi_Q V\cdot$

According to this definition, $\Pi_Q U_h$ is a weak approximate solution of $(1.1)-(1.2)$. For sake of simplicity of notations and clarity of the presentation, we denote in the sequel this weak approximate solution u_h instead of $\Pi_Q U_h$.

Note that U_L represents $u_{i,\,j}$ or $u_{i+\frac{1}{2},\,j+\frac{1}{2}}$ depending on whether x_L is a cell center or a corner point in the primary grid.

4.3.2 A stability result

The stability analysis of the weak approximate solution is the main subject within this part of the presentation. The main ingredient for this analysis is a discrete version of the Poincaré inequality that reads as follows.

PROPOSITION 4.4 (discrete version of Poincaré inequality)
There exists a strictly positive number P such that

$$\|v\|_{L^2(\Omega)} \leq P \|v\|_{1,h} \qquad \forall\, v \in \mathbf{E}(\mathcal{L})$$

where we have set

$$\|v\|_{L^2(\Omega)} = \left(\int_\Omega v^2 dx \right)^{\frac{1}{2}}$$

Proof. Let us define the two following functions almost everywhere in Ω by:
$E(x) = L$, with $L \in \mathcal{L}$, $x \in Int(L) \equiv$ interior of L

$$\chi_s(x) = \left\{ \begin{array}{l} 1 \; if \quad s \in D_{E(x)} \\ 0 \;\; \text{otherwise} \end{array} \right. \qquad \text{with} \quad s \in S$$

where D_L, for $L \in \mathcal{L}$, denotes the semi-line defined by its origin x_L (center of the element L) and the vector $d = (1,0)^T$. We set for all $v \in \mathbf{E}(\mathcal{L})$:
$[\Delta_{s,d} v] = |v_L - v_K|$ if $s \in \Gamma_K \cap \Gamma_L$ and d is the direction of the line $(x_L x_K)$.
Let $v \in \mathbf{E}(\mathcal{L})$ and $L \in \mathcal{L}$. We consider $x \in \Omega$ such that $x \in Int(L)$. It is clear that

$$v(x) = |v_L| \leq \sum_{s\in S} \chi_s(x) [\Delta_{s,d} v]$$

We deduce, thanks to Cauchy-Schwarz inequality,

$$|v(x)|^2 \leq |v_L|^2 \leq \left(\sum_{s\in S} h \chi_s(x) \right) \left(\sum_{s\in S} \frac{\chi_s(x)}{h} [\Delta_{s,d} v]^2 \right) \tag{4.19}$$

Remarking that

$$\sum_{s\in S}\chi_s(x)\,h \leq diam\,(\Omega) \equiv \text{diameter of } \Omega$$

one obtains after integration in Ω of the both sides of (4.19) :

$$\|v\|^2_{L^2(\Omega)} \leq diam\,(\Omega)\sum_{s\in S}\left\{\frac{1}{h}\left[\Delta_{s,d}v\right]^2\int_\Omega \chi_s(x)\,dx\right\} \tag{4.20}$$

Since

$$\forall s \in S \qquad \int_\Omega \chi_s(x)\,dx \leq h\ diam\,(\Omega)$$

one deduces from (4.20) that

$$\|v\|^2_{L^2(\Omega)} \leq \left[diam\,(\Omega)\right]^2\sum_{s\in S}\left[\Delta_{s,d}v\right]^2 \leq \left[diam\,(\Omega)\right]^2\|v\|^2_{1,h}$$

This ends the proof of Proposition 4.4 · ■

Proposition 4.5 (Stability result)
The weak approximate solution u_h of the diffusion problem (1.1)-(1.2) satisfies the following inequality:

$$\|u_h\|_{1,h} \leq C\,\|f\|_{L^2(\Omega)}$$

where C is a strictly positive real number not depending on the spatial discretization .

Proof. Multiplying (4.6) and (4.7) by $u_{i,j}$ and $u_{i+\frac{1}{2},j+\frac{1}{2}}$ respectively, and besides summing for $i,j \in \{1,2,...,N\}$ and for $i,j \in \{1,2,...,N-1\}$ respectively yields (with utilization of notation (4.11) and matrix formulation (4.10)) :

$$\left[U_{cc}\quad U_{vc}\right]\begin{bmatrix} A & B \\ B^T & C\end{bmatrix}\begin{bmatrix} U_{cc} \\ U_{vc}\end{bmatrix} = \left[U_{cc}\quad U_{vc}\right]\begin{bmatrix} F_{cc} \\ F_{vc}\end{bmatrix} \tag{4.21}$$

Let us recall that

$$F_{cc} = \left\{\int_{K_{i,j}} f\,dx\right\}_{1\leq i,j\leq N} \qquad \text{and} \qquad F_{vc} = \left\{\int_{K_{i+\frac{1}{2},\ j+\frac{1}{2}}} f\,dx\right\}_{1\leq i,j\leq N} \tag{4.22}$$

From the proof of *Proposition* 4.2 we know that the left hand side of (4.21) satisfies the following inequality :

$$\gamma\,\|u_h\|^2_{1,h} \leq \left[U_{cc}\quad U_{vc}\right]\begin{bmatrix} A & B \\ B^T & C\end{bmatrix}\begin{bmatrix} U_{cc} \\ U_{vc}\end{bmatrix} \tag{4.23}$$

In addition, the right hand side of (4.21) obeys to the following relation

$$\left|\left[U_{cc}\quad U_{vc}\right]\begin{bmatrix} F_{cc} \\ F_{vc}\end{bmatrix}\right| \leq \sum_{c\in C}\sum_{s\in A(c)}\left[\int_{Kcs}\left(|f|\,|U_s + U_c|\right)dx\right] \equiv RHS \tag{4.24}$$

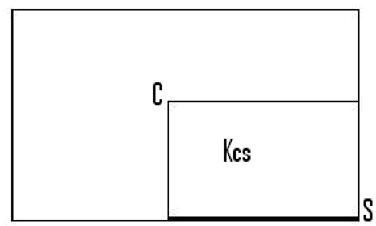


Figure 8: A cell of the primary mesh including a sub-cell K_{cs}

where C is the set of cell centers of the primary mesh, $A(c)$ the set of vertices of the cell centered on c, with $c \in C$, K_{cs} the quarter of a cell from the primary mesh, containing the points c and s, with $c \in C$ and $s \in A(c)$ (*see Fig* 8 above), u_s value of u_h in the element of $\mathcal{L}$ centered on s, idem for u_c.

An application of Cauchy-Schwarz inequality to the integral term of (4.24) yields

$$\sum_{c\in C}\sum_{s\in A(c)}\left[\int_{Kcs}\left(|f|\,|U_s+U_c|\right)dx\right] \leq \sum_{c\in C}\sum_{s\in A(c)}\left[\frac{h}{2}\left(|U_s|+|U_c|\right)\left(\int_{K_{cs}}f^2dx\right)^{\frac{1}{2}}\right]$$

A double application of discrete Cauchy-Schwarz inequality leads to

$$RHS \leq 2\left[\sum_{c\in C}\sum_{s\in A(c)}\int_{K_{cs}}f^2dx\right]^{\frac{1}{2}}\left[\sum_{c\in C}\sum_{s\in A(c)}\frac{h^2}{8}\left(U_s^2+U_c^2\right)\right]^{\frac{1}{2}} \tag{4.25}$$

Remarking that

$$\sum_{c\in C}\sum_{s\in A(c)}\frac{h^2}{8}\left(U_s^2+U_c^2\right) = \int_\Omega |u_h|^2\,dx$$

and thanks to (4.23)-(4.25) and *Proposition* 4.4, the proof of *Proposition* 4.5 is ended · ■

4.3.3 Error estimates

When accounting with the truncation error, the equations (4.2)-(4.5) are transformed as follows :

$$\begin{aligned} &D_{21}\left[\varphi_{i-\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\right]+D_{22}\left[\varphi_{i,j}-\varphi_{i,j+1}\right]+D_{21}\left[\varphi_{i+\frac{1}{2},j-\frac{1}{2}}-\varphi_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\ &+D_{22}\left[\varphi_{i,j}-\varphi_{i,j-1}\right]+D_{12}\left[\varphi_{i+\frac{1}{2},j-\frac{1}{2}}-\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\right]+D_{11}\left[\varphi_{i,j}-\varphi_{i+1,j}\right]\\ &+D_{12}\left[\varphi_{i-\frac{1}{2},j+\frac{1}{2}}-\varphi_{i-\frac{1}{2},j-\frac{1}{2}}\right]+D_{11}\left[\varphi_{i,j}-\varphi_{i-1,j}\right]\\ &=\int_{K_{ij}}f(x)dx \quad + \quad \sum_{e\in E_{i,j}} hR^e_{i,j} \qquad \forall \quad 1\leq\ i,\ j\leq\ N \end{aligned} \tag{4.26}$$

$$D_{21}\left[\varphi_{i,j+1}-\varphi_{i+1,j+1}\right]+D_{22}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j+\frac{3}{2}}\right]+$$

$$D_{21}\left[\varphi_{i+1,j}-\varphi_{i,j}\right]+D_{22}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j-\frac{1}{2}}\right]$$

$$+D_{12}\left[\varphi_{i+1,j}-\varphi_{i+1,j+1}\right]+D_{11}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{3}{2},j+\frac{1}{2}}\right]$$

$$+D_{12}\left[\varphi_{i,j+1}-\varphi_{i,j}\right]+D_{11}\left[\varphi_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i-\frac{1}{2},j+\frac{1}{2}}\right]$$

$$=\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} f(x)dx \quad + \quad \sum_{e\in E_{i+\frac{1}{2},j+\frac{1}{2}}} hR^e_{i+\frac{1}{2},j+\frac{1}{2}} \quad \forall \quad 1\leq\ i,\ j\leq\ N\ -1 \tag{4.27}$$

with the following discrete boundary conditions :

$$\varphi_{i+\frac{1}{2},\frac{1}{2}}=\varphi_{i+\frac{1}{2},N+\frac{1}{2}}=\varphi_{\frac{1}{2},j+\frac{1}{2}}=\varphi_{N+\frac{1}{2},j+\frac{1}{2}}=0 \quad \forall \quad 0\leq\ i,\ j\leq\ N \tag{4.28}$$

$$\varphi_{i,0}=\varphi_{0,j}=\varphi_{i,N+1}=\varphi_{N+1,j}=0 \qquad \forall \quad 1\leq\ i,\ j\leq\ N \tag{4.29}$$

where $E_{i,j}$ and $E_{i+\frac{1}{2},j+\frac{1}{2}}$ are sets of edges associated respectively with $K_{i,j}$ and $K_{i+\frac{1}{2},j+\frac{1}{2}}$, and where $R^e_{i,j}$ and $R^e_{i+\frac{1}{2},j+\frac{1}{2}}$ denote the truncation error associated with the approximation of the flux over the edges $e_{i,j}$ and $e_{i+\frac{1}{2},j+\frac{1}{2}}$ respectively. Moreover, under the assumption $\varphi\in C^3\left(\overline{\Omega}\right)$the truncation error satisfy the following inequalities :

$$\left|R^e_{i,j}\ \right|\leq Ch^2 \qquad\text{and}\qquad \left|R^e_{i+\frac{1}{2},j+\frac{1}{2}}\ \right|\leq Ch^2 \tag{4.30}$$

in what follows, the notation R^e_K will be utilized to denote the truncation error for the approximation of the flux over the edge e_K of any control volume K. Thanks to the conservativity property of the proposed finite volume formulation, we have

$$R^e_K \quad + \quad R^e_I=0 \tag{4.31}$$

where K and I are two adjacent control volumes such that $e=\Gamma_K\cap\Gamma_I$. Let us define a function ε_h almost everywhere in $\mathbb{R}^2$ the following way :

$$\varepsilon_h\left(x\right)=\begin{cases}\varepsilon_L & \text{if } \ x\in Int(L)\\ \quad 0 & \text{elsewhere}\end{cases} \qquad \text{with} \quad L\in\mathcal{L} \tag{4.32}$$

where we have set $\varepsilon_L=\varphi_L-u_L$ for all $L\in\mathcal{L}$. Note that the element L of the secondary mesh $\mathcal{L}$ is necessary centered on a point whose cartesian coordinates are of the form $\left(x^i_1,x^j_2\right)$ or $\left(x_1^{i+\frac{1}{2}},x_2^{j+\frac{1}{2}}\right)$. ε_L is the generic name for $\varepsilon_{i,j}$ or $\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}$.

Remark 3 From the relation (4.32) we see that the function is actually in $\mathbf{E}(\mathcal{L})$. This function expresses the error in some sense (i.e. the difference between the exact and the weak approximate solution u_h) and certain estimates of this error are given in what follows ·

We immediately are going to show that the following quantities $\{\varepsilon_{i,\,j}\}_{1\leq\, i,\ j\leq\ N}$ and $\left\{\varepsilon_{i+\frac{1}{2},\,j+\frac{1}{2}}\right\}_{1\leq\, i,\ j\leq\ N-1}$ are a solution of a discrete problem of the form (4.6) – (4.9). Subtracting (4.6) from (4.26) and (4.7) from (4.27), and reordering the terms yields :

$$
\begin{aligned}
&D_{21}\left[\varepsilon_{i-\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}\right]\ +\ D_{22}\left[\varepsilon_{i,j}-\varepsilon_{i,j+1}\right]\ +\ D_{21}\left[\varepsilon_{i+\frac{1}{2},j-\frac{1}{2}}-\varepsilon_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\
&+D_{22}\left[\varepsilon_{i,j}-\varepsilon_{i,j-1}\right]\ +\ D_{12}\left[\varepsilon_{i+\frac{1}{2},j-\frac{1}{2}}-\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}\right]\ +\ D_{11}\left[\varepsilon_{i,j}-\varepsilon_{i+1,j}\right]\\
&+D_{12}\left[\varepsilon_{i-\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i-\frac{1}{2},j-\frac{1}{2}}\right]\ +\ D_{11}\left[\varepsilon_{i,j}-\varepsilon_{i-1,j}\right]\\
&=\sum_{e\in E_{i,j}} hR^{e}_{i,j}\qquad\qquad \forall\quad 1\leq\ i,\ j\leq\ N
\end{aligned}
\tag{4.33}
$$

$$
\begin{aligned}
&D_{21}\left[\varepsilon_{i,j+1}-\varepsilon_{i+1,j+1}\right]+D_{22}\left[\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i+\frac{1}{2},j+\frac{3}{2}}\right]\\
&+D_{22}\left[\varepsilon_{i+1,j}-\varepsilon_{i,j}\right]\ +\ D_{22}\left[\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i+\frac{1}{2},j-\frac{1}{2}}\right]\\
&+D_{12}\left[\varepsilon_{i+1,j}-\varepsilon_{i+1,j+1}\right]\ +\ D_{11}\left[\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i+\frac{3}{2},j+\frac{1}{2}}\right]\\
&+D_{12}\left[\varepsilon_{i,j+1}-\varepsilon_{i,j}\right]\ +\ D_{11}\left[\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}-\varepsilon_{i-\frac{1}{2},j+\frac{1}{2}}\right]\\
&=\sum_{e\in E_{i+\frac{1}{2},j+\frac{1}{2}}} hR^{e}_{i+\frac{1}{2},j+\frac{1}{2}}\qquad\qquad \forall\quad 1\leq\ i,\ j\leq\ N\ -1
\end{aligned}
\tag{4.34}
$$

with, thanks to (4.8), (4.9), (4.28) and (4.29),

$$\varepsilon_{i+\frac{1}{2},\frac{1}{2}}=\varepsilon_{i+\frac{1}{2},N+\frac{1}{2}}=\varepsilon_{\frac{1}{2},j+\frac{1}{2}}=\varepsilon_{N+\frac{1}{2},j+\frac{1}{2}}=0\qquad\forall\quad 0\leq\ i,\ j\leq\ N \tag{4.35}$$

$$\varepsilon_{i,0}=\varepsilon_{0,j}=\varepsilon_{i,N+1}=\varepsilon_{N+1,j}=0\qquad\forall\quad 1\leq\ i,\ j\leq\ N \tag{4.36}$$

Multiplying (4.33) and (4.34) by $\varepsilon_{i,j}$ and $\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}$ respectively, summing over i,j and reordering the terms of the left hand side after summation side by side of the two final equations, leads to the following inequality thanks to (1.3) :

$$\begin{aligned}
\gamma \left\|\varepsilon_h\right\|_{1,h}^2 &\leq \sum_{1\leq\ i,\ j\leq\ N} \left(h\ \varepsilon_{i,j} \sum_{e\in E_{i,j}} R_{i,j}^e \right) \\
&+ \sum_{1\leq\ i,\ j\leq\ N-1} h\ \varepsilon_{i+\frac{1}{2},j+\frac{1}{2}} \left(\sum_{e\in E_{i+\frac{1}{2},j+\frac{1}{2}}} R_{i+\frac{1}{2},j+\frac{1}{2}}^e \right) \\
&\leq h \sum_{1\leq\ i\leq\ N,\ 1\leq\ \ j\leq\ N} a_{i,j} \left[\left|\varepsilon_{i,j} - \varepsilon_{i,j+1}\right| + \left|\varepsilon_{i-\frac{1}{2},j+\frac{1}{2}} - \varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}\right| \right] \\
&+ h \sum_{0\leq\ i\leq\ N,\ 1\leq\ \ j\leq\ N} b_{i,j} \left[\left|\varepsilon_{i,j} - \varepsilon_{i+1,j}\right| + \left|\varepsilon_{i-\frac{1}{2},j+\frac{1}{2}} - \varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}\right| \right] \\
&\leq \text{(double application of Cauchy-Schwarz inequality)} \\
&\leq 2h \left[\sum_{1\leq\ i\leq\ N,\ 0\leq\ \ j\leq\ N} a_{i,j}^2 + \sum_{0\leq\ i\leq\ N,\ 1\leq\ \ j\leq\ N} b_{i,j}^2 \right]^{\frac{1}{2}} \left\|\varepsilon_h\right\|_{1,h}
\end{aligned}$$

where we have set, for $1 \leq\ i \leq\ N$ and $0 \leq\ \ j \leq\ N$:

$$a_{i,j} = \max \left\{ R_{i,j}\ ,\ R_{i-\frac{1}{2},j+\frac{1}{2}} \right\}$$

with

$$R_{i,j} = \max_e \left|R_{i,j}^e\right|,\ R_{i-\frac{1}{2},j+\frac{1}{2}} = \max_e \left|R_{i-\frac{1}{2},j+\frac{1}{2}}^e\right|$$

and for $0 \leq\ i \leq\ N$ and $1 \leq\ \ j \leq\ N$:

$$b_{i,j} = \max \left\{ R_{i,j}\ ,\ R_{i+\frac{1}{2},j-\frac{1}{2}} \right\}$$

with

$$R_{i,j} = \max_e \left|R_{i,j}^e\right|,\ R_{i+\frac{1}{2},j-\frac{1}{2}} = \max_e \left|R_{i+\frac{1}{2},j-\frac{1}{2}}^e\right|$$

Therefore, we deduce, thanks to (4.30), that for $\varphi \in C^3\left(\overline{\Omega}\right)$ we have

$$\left\|\varepsilon_h\right\|_{1,h} \leq \Lambda h^2 \tag{4.37}$$

where Λ is a positive real number depending exclusively on φ, Ω and γ.

Let us now investigate an error estimate for the $L^\infty - norm$ defined over the space $\mathbf{E}(\mathcal{L})$ by:

$$\left\|v_h\right\|_{L^\infty(\Omega)} = \max_L \left|v_L\right| \quad \text{or, which is equivalent, } \left\|v_h\right\|_{L^\infty(\Omega)} = \max_{1\leq i,j\leq N} \left|v_{i,j}\right|\ .$$

Since $\varepsilon_{0,j} = 0$ (see relation (4.36)) it is obvious that

$$\varepsilon_{i,j} = -\varepsilon_{0,j} + \varepsilon_{1,j}\ - \varepsilon_{1,j} + \varepsilon_{2,j} - \varepsilon_{2,j} + ... + \varepsilon_{i-1,j} - \varepsilon_{i-1,j} + \varepsilon_{i,j} \qquad \forall 1 \leq i,j \leq N$$

Then, thanks to Minkowski inequality, one deduces that

$$\begin{aligned} |\varepsilon_{i,j}| &\leq |-\varepsilon_{0,j}+\varepsilon_{1,j}| + |-\varepsilon_{1,j}+\varepsilon_{2,j}| + ... + |-\varepsilon_{i-1,j}+\varepsilon_{i,j}| \\ &\leq \sum_{k=0}^{N} |\varepsilon_{k+1,j}-\varepsilon_{i,j}| \qquad \forall 1 \leq i,j \leq N \end{aligned}$$

with $\varepsilon_{N+1,j} = 0$ (see relation (4.36)). Applying Cauchy-Schwarz inequality leads to:

$$\max_{1\leq i,j\leq N} |\varepsilon_{i,j}| \leq \sqrt{2}h^{-\frac{1}{2}} \left\|\varepsilon_h\right\|_{1,h}$$

It then follows, utilizing (4.37), that

$$\left\|\varepsilon_h\right\|_{L^\infty(\Omega)} \leq \sqrt{2}\Lambda h^{\frac{3}{2}} \tag{4.38}$$

Let us summarize these error estimates (i.e. (4.37) and (4.38)) in the following assertion.

Theorem 4.6 (Error estimates in following norms: $\mathbf{L}^\infty(\Omega)$ and $\|.\|_{1,h}$)
Assume that the diffusion tensor D in the Diffusion problem (1.1)-(1.2) is a positive definite full matrix with constant coefficients (not necessary symmetric). Assume also that the unique variational solution φ of (1.1)-(1.2) satisfies $\varphi \in C^3\left(\overline{\Omega}\right)$ and let us consider the space $\mathbf{E}(\mathcal{L})$ made of functions v defined almost everywhere in $\mathbb{R}^2$ such that v is constant in each element of the mesh $\mathcal{L}$ (see Fig 7 for the definition of $\mathcal{L}$). Define φ_h (where h is the size of the primary mesh associated with the finite volume formulation of the problem (1.1)-(1.2)) as a function of $\mathbf{E}(\mathcal{L})$ satisfying the following property :

$$\varphi_{h_{|L}}(x) = \varphi_L \equiv \text{value of } \varphi \text{ at the center of } L, \text{ for all } L \in \mathcal{L}$$

and (of course) zero elsewhere.

Then, the function $\varepsilon_h = \varphi_h - u_h$, where ε_h is defined by (4.32) and $u_h = \Pi_Q U_h$, satisfies the following inequalities:

$$\begin{array}{lrcl} (i) & \left\|\varepsilon_h\right\|_{1,h} & \leq & \Lambda h^2 \\ (ii) & \left\|\varepsilon_h\right\|_{L^\infty(\Omega)} & \leq & \sqrt{2}\Lambda h^{\frac{3}{2}} \end{array}$$

where Λ is a strictly positive real number depending exclusively on φ, Ω and γ, and where P is a strictly positive constant involved in the discrete version of Poincaré inequality (*Proposition* 4.3)·

Corollary 4.7 (Error estimate in $L^2(\Omega) - norm$)
ε_h satisfies the following inequality

$$\left\|\varepsilon_h\right\|_{L^2(\Omega)} \leq P\Lambda h^2\cdot$$

Proof. This inequality follows obviously from the inequality (i) of Theorem 4.6 and Proposition 4.4· ■

5 Numerical experiments and comparison with MPFA O-Method.

The most popular MPFA method is the MPFA O-method which is a finite-volume based strategy to compute diffusion in anisotropic non-homogeneous media [ABBM 96a]. Its main feature is that, after calculation of local transmissivities in order to insure the continuity of flux on grid blocks interfaces, it leads to a discrete problem where unknowns are cell-centered. We present two groups of numerical tests based upon Dirichlet boundary value problems. In the first group, the validation of theoretical estimates (i), (ii) and (iii) given in theorem 4.1 is our main objective. The second group of numerical tests is devoted to the comparison of our MPFA finite volume method with MPFA O-method presented in [ABBM 96a].

Group 1: We consider two test Dirichlet problems on the square domain $\Omega =]0,1[^2$. In each problem, we select a uniform full diffusion tensor D and the exact solution $\varphi(x_1, x_2)$. Then after, we calculate the corresponding right hand side in the continuity equation (1.1). In order to validate the theoretical estimates given in theorem 4.1, the approximate solution considered is a piece-wise constant function (for details see section 3.2). The error ε_h is performed in the following norms:

$$\|\varepsilon_h\|_{1,h} = \left[\sum_{\substack{K,L such that \\ \exists s\in S\ \Gamma_K\cap\Gamma_L=\{s\}}} |\varepsilon_L - \varepsilon_K|^2 \right]^{\frac{1}{2}}, \quad \text{where} \quad \varepsilon_L = \varphi_L - u_L$$

$$L^\infty - Error = \max_{i,j=1,\dots,N} |\varphi_{ij} - u_{ij}| \qquad \text{and} \qquad L^2 - Error = \left[\int_\Omega (\varphi - u_h)^2 \, dx \right]^{\frac{1}{2}}$$

Group2: We consider two test Dirichlet problems on the square domain $\Omega =]0,1[^2$. In each problem, we select a full diffusion tensor $D(x_1, x_2)$, the exact solution $\varphi(x_1, x_2)$ and calculate the corresponding right hand side in the continuity equation (1.1). In order to compare our MPFA finite volume Method with the MPFA O-method, the approximate solution considered is a piece-wise linear function (for details see subsection 3.2). The difference between the exact solution φ and the computed solution is given in these norms:

$$L^\infty - Error = \max_{i,j=1,\dots,N} |\varphi_{ij} - u_{ij}| \qquad \text{and} \qquad L^2 - Error = \left[\int_\Omega (\varphi - u_h)^2 \, dx \right]^{\frac{1}{2}}$$

The grid used here is a non rectangular quadrilateral grid. In other to compute the numerical integral of the right hand-side of the balance (1.1) equation, each element of the primary mesh is divided in two triangles. Then, in each triangle, numerical integration is performed using the formulae presented in [RNVR 04]. Let us mention that in our previous work [NN 05a], the same problems were studied and a different numerical integration formula was applied. The results obtained here are significantly better than the one presented previously.

5.1 Group 1

5.1.1 First test problem

$$-div(D\ grad\ u) = f \qquad \text{in} \quad]0,1[^2 \qquad \text{with} \quad D = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$$

The exact solution is

$$\varphi\left(x_1, x_2\right) = x_1\left(x_1 - 10\right) x_2\left(x_2 - 10\right)$$

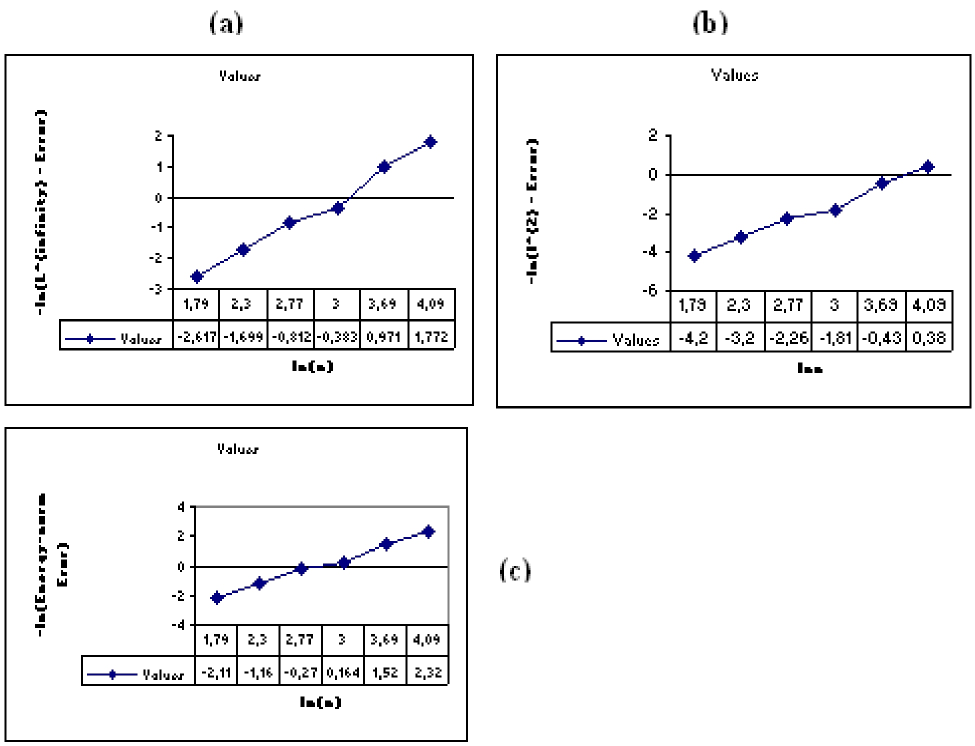


Figure 9: (a): Log of L^{∞}-error for the proposed method, with the convergence rate which is equal to 1.91•
(b): Log of L^2-error for the proposed method,with the convergence rate which is equal to 1.99•
(c): Log of $\|\cdot\|_{1,h}$-error for the proposed method, with the convergence rate which is equal to 1.93•

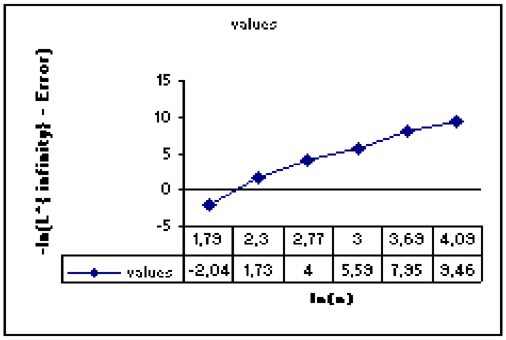
5.1.2 Second test problem

$$-div(D\ grad\ u) = f \qquad \text{in} \quad]0,1[^2 \qquad \text{with} \quad D = \begin{bmatrix} 1 & 10 \\ 10 & 1000 \end{bmatrix}$$

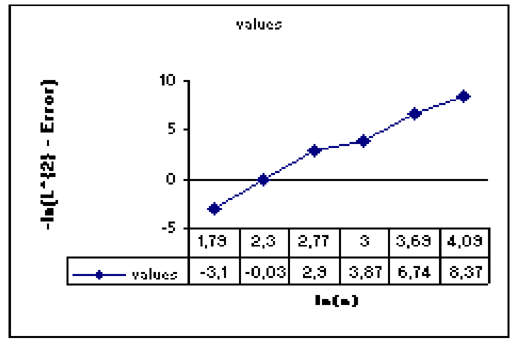
The exact solution is

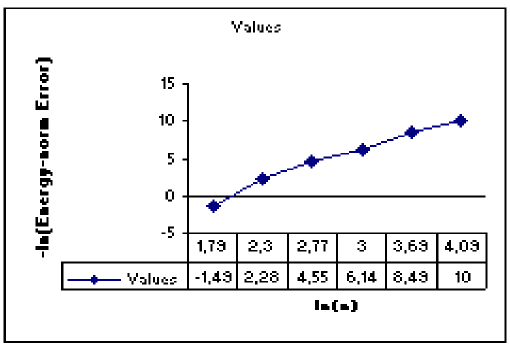
$$\varphi\left(x_1, x_2\right) = \sin(\pi x_1)\sin(\pi x_2)$$

(a)



(b)





(c)

Figure 10: (a): Log of L^{∞}-error for the proposed method, with the convergence rate which is equal to 4.85 •
(b): Log of L^2-error for the proposed method, with the convergence rate which is equal to 4.93•
(c): Log of $\|\cdot\|_{1,h}$-error for the proposed method, with the convergence rate which is equal to 4.86 •

5.2 Group 2

5.2.1 First test problem (extracted from [Y 96])

$-div\,[D\ grad\ u] = f \qquad in \quad]0,1[^2$ with $D = D\,(x_1,x_2) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ if $x_1 \le 0.5$,

and

$D = D\,(x_1,x_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ otherwise.

The exact solution reads

$$\varphi\,(x_1,x_2) = \begin{cases} x_1x_2 & \text{if} \quad x_1 \le 0.5 \\ \\ x_1x_2 + (x_1 - 0.5)\,(x_2 + 0.5) & \text{if} \quad x_1 \succ 0.5 \end{cases}$$

Applying the proposed Method with 6×6 primary grid cells, the L^{∞}-error is $1.19E-8$. Refining the mesh does not lead to better results. With 60×60 grid cells, the

L^∞-error for the O-Method is $3.83E-5$. So, this error remains much greater than the L^∞-error for the proposed method with 6×6 primary grid cells.

5.2.2 second test problem

$$-div(D\ grad\ u) = f \qquad \text{in} \quad]0,1[^2 \qquad \text{with} \quad D = \begin{bmatrix} 1 & 10 \\ 10 & 1000 \end{bmatrix}$$

The exact solution is $\varphi(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$

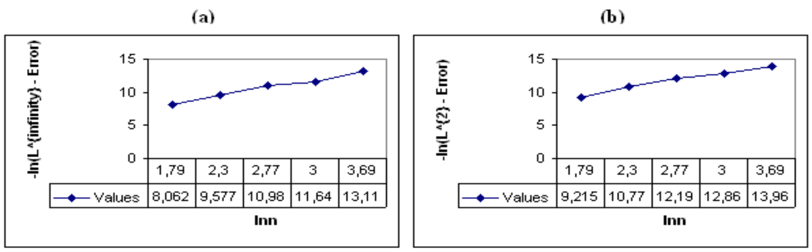


Figure 11: (a): Log of L^∞-error for the proposed method, with the convergence rate equal which is equal to 2.69 •
(b): Log of L^2 -error for the proposed method, with the convergence rate which is equal to 2.55 •

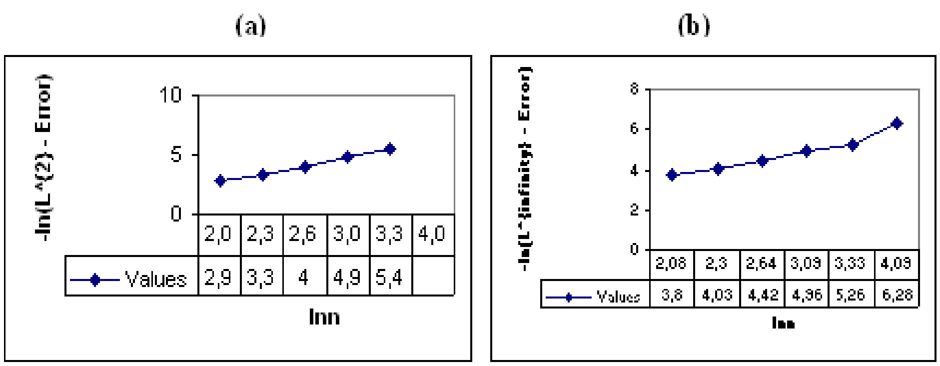


Figure 12: (a): Log of L^∞-error for the O-method, with the convergence rate which is equal to 1.22.

Log of L^2-error for the O-method, with the convergence rate which is equal to 2.

5.3 Concluding remark

The numerical experiments performed in the case of anisotropic homogeneous media, with non necessary symmetric tensor (test problems of group 1), confirm clearly the theoretical results (given in the Theorem above) which asserts that the proposed MPFA method leads to a super convergence in L^∞-norm. In another hand, it leads to a quadratic convergence in the L^2-norm and the discrete energy norm $\|\cdot\|_{1,h}$. Regarding anisotropic nonhomogeneous media, theoretical investigations are going on in view to obtain stability and error estimates results. However, numerical simulations performed in such media (see test problems of group 2), show the convergence of the proposed method and its good behavior in comparison with the O-method.

6 Conclusions and perspectives

We have presented in this work a new MPFA method for addressing flow problems in anisotropic non-homogeneous media. This method has displayed a large capability for computing, with a satisfactory accuracy, the flows within homogeneous and non-homogeneous media involving strong anisotropies (see numerical experiments of section 5). In the homogeneous frame-work, stability and error estimates have been proven for the proposed method in L^∞-norm, L^2-norm and the discrete energy norm $\|\cdot\|_{1,h}$. These results have been confirmed by numerical experiments performed with various examples. Some numerical experiments have been performed with examples extracted from the literature. In this connection, the proposed MPFA method turns out to be more performant than the so-called MPFA O-method widely used in simulation of flows in geologically complex reservoir. It is worth mentioning that when the flow coefficient is a constant full matrix, not necessarily symmetric but positive definite (corresponding physically to an anisotropic homogeneous medium), the proposed MPFA method leads to a symmetric positive definite discrete system. Moreover if the diffusion coefficient is diagonal this discrete system is actually made of two independent linear system corresponding to the well-known cell-centered and vertex-centered finite volume methods.

Theoretical investigations regarding the stability and the convergence of this method when the flow coefficient is a spatially varying full matrix is our research project.

Acknowledgement: The authors would like to thank the anonymous referees for their pertinent remarks and useful suggestions. They would also like to express their gratitude to A.J. Kinfack and P.E. Njiki Njiki for their technical assistance regarding the typing.

7 Bibliography

[ABBM 96a] Aavatsmark I., Barkve T., BOE O., Mannseth T. (1996), A Class of Discretization Methods for Structured and Unstructured Grids in
Anisotropic, Inhomogeneous Media, 5th European Conference on the Mathematics of Oil Recovery, Sept.1996.

[ABBM 98] Aavatsmark I., Barkve T., BOE O., Mannseth T. (1998), Discretization on unstructured grids for inhomogeneous,
anisotropic media. part I: derivation of the methods, Siam, Vol 19, N° 5,1998, pp. 1700-1716

[ABBM 96b] Aavatsmark I., Barkve T., BOE O., Mannseth T. (1996), Discretization on Non-Orthogonal, Quadrilateral Grids for Inhomogeneous,
Anisotropic Media. J. Comput. Phys. 127, 1996, pp. 2-14

[B 83] Brézis H. (1983), Analyse fonctionnelle , théorie et applications, Ed. Masson

[BLP 78] Bensoussan A., Lions J.L. and Papanicolaou (1978), Asymptotic analysis for periodic structures, Amsterdam (North-Holland) 1978

[CCJ 87] A finite element simulator for incompressible two-phase flow, Transport in Porous Media 2, (1987) pp. 465-478.

[DGJ 03] Duq, Gunzburger M.D. and Ju L. (2003), Voronoï-based finite volume methods, optimal voronoï meshes, and PDE on the sphere, Comput. Methods App. Mech. Engrg. (2003) pp.3933-3957.

[D 05] Durlofsky L. J. (2005), Upscaling and Gridding of Geologically Complex Systems, Department of Petroleum Engineering, Stanford University Chevron Texaco E & P Technology Company

[EAAKN 04] Eigestad G.T., Aadland T., Aavastmark I., Klausen R.A., Nordbotten J.M., Recent advances for MPFA methods., Ecmor IX , 2004.

[EGH 00] Eymard R., Gallouët T. and Herbin R. (2000), Finite volume methods, Handbook of Numerical Analysis, Eds : P.G. Ciarlet and J.L. Lions, North Holland, 7 (2000), pp. 713-1020.

[EGH 02] Eymard R., Gallouët T. and Herbin R. (2002), A finite volume scheme for anisotropic diffusion problems, C.R. Acad. Sci. Paris (Note presented by P.G. Ciarlet)

[EK 05] Eigestad G.T. and Klausen R.A.(2005), On the convergence of the multi-point flux approximation O-method; Numerical experiments for discontinuous permeability, Numer. Methods Partial Diff. Eqns. 21(2005), 1079-1098

[H 02] Hoteit H. (2002), Simulation d'écoulements et de transports de polluants en milieu poreux : Application à la modélisation de la sûreté des dépôts de déchets radioactifs , PhD thesis, Université de Rennes 1 (France).

[K 03], Klausen R. A. (2003), Convergence Of Multi-Point Flux Approximations On Quadrilateral Grids, (Report, University of Oslo).

[LDLC 98] Lee S.H., Durlofsky L.J., Lough M.F. and Chen W.H. (1998), "Finite Difference Simulation of Geologically Complex Reservoirs With Tensor Permeabilities", SPE Reservoir Evaluation & Engineering, December 1998

[M 03] Mishev I.D. (2003), "Analysis of a new mixed finite volume method", Computational methods in applied mathematics, vol. 3(2003), N°1, pp.189-201

[N 94] Njifenjou, A. (1994), Expression en termes d'énergie pour la perméabilité absolue effective. Application au calcul numérique d'écoulements diphasiques en milieu poreux, Revue de l'Institut Français du Pétrole, vol.49, N°4, Juillet-Août 1994

[NM 01] Njifenjou A. and Moukouop Nguena I. (2001), "Traitement des anisotropies de perméabilité en simulation d'écoulement en milieu poreux par les volumes finis", Colloque SIGE, Douala (Cameroun), pp. 245-259.

[NN 05a] Nguena I.Moukouop. and Njifenjou A. (2005), "A new finite volume formulation for diffusion problems in anisotropic non-homogeneous media", 4th International Symposium on Finite Volume for Complex Applications, Marraketch (Morocco), 2005, July 4-8, pp. 435-446

[NN 05b] Njifenjou A. and Nguena I.Moukouop. (2005), "A new finite volume method for computing flow in geologically complex reservoirs", Trends in Numerical and Physical Modeling for Industrial Multiphase Flows, Cargèse (Corsica), 2005, September 19-23.

[NM 05] Njifenjou A. and Moukouop Nguena. I. (2005), "A new finite volume method for flow problems in non-homogeneous anisotropic media : Derivation of the scheme", Séminaire/Atelier international sur la pollution des sols et des aquifères, Brazzaville (Congo), 04-09 Juillet 2005.

[NMK 04] Njifenjou A., Mampassi B. et Kinfack A.J. (2004), Modélisation d'écoulement dans une nappe phréatique en relation avec des lacs et des rivières, International Conference of RAMAD on Modelling and Simulation in Environment, Cheikh Anta Diop University, Dakar, August 02 - 07, 2004

[R 05] Renard P. (2005), Averaging methods for permeability fields, Development, Protection, Management and Sequestration of Subsurface. Fluid Flow and Transport in Porous and Fractured Media, Summer School, Cargèse (France) 2005, july 27

[RNVR 04] Rathod H.T., Nagaraja K.V., Venkatesudu B., and Ramesh N. L. (1984), Gauss Legendre quadrature over a triangle, J. Indian Inst. Sci., Sept.–Oct. 2004, 84, 183–188

[RT 88] Raviart P.A. et Thomas J.M. (1988), Introduction à l'analyse numrique des équations aux dérivées partielles, Collection Mathématiques Appliquées pour la Maîtrise, under the direction of P.G. Ciarlet et J.L. Lions. Editor : Masson.

[SP 80] Sanchez-Palencia E. (1980), Non-homogeneous media and vibration theory, Lecture notes in Physics, Springer-Verlag.

[VA 96] Verma S. and Aziz K. (1996), Two- and three-dimensional flexible grids for reservoir simulation, 5th European Conference on the Mathematics of Oil Recovery, Leoben, Austria, 3-6 sept 1996.

[Y 96] Yotov I. (1996), Mixed Finite Element Methods for Flow in Porous Media, PhD thesis, Rice University (USA)