Gradient schemes for the Stefan problem

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Abstract

We show in this paper that the gradient schemes (which encompass a large family of discrete schemes) may be used for the approximation of the Stefan problem $\partial_t \bar{u} - \Delta\zeta(\bar{u}) = f$. The convergence of the gradient schemes to the continuous solution of the problem is proved thanks to the following steps. First, estimates show (up to a subsequence) the weak convergence to some function u of the discrete function approximating $\bar{u}$. Then Alt-Luckhaus' method, relying on the study of the translations with respect to time of the discrete solutions, is used to prove that the discrete function approximating $\zeta(\bar{u})$ is strongly convergent (up to a subsequence) to some continuous function χ. Thanks to Minty's trick, we show that $\chi = \zeta(u)$. A convergence study then shows that u is then a weak solution of the problem, and a uniqueness result, given here for fitting with the precise hypothesis on the geometric domain, enables to conclude that $u = \bar{u}$. This convergence result is illustrated by some numerical examples using the Vertex Approximate Gradient scheme.

Key words : Stefan problem, gradient schemes, uniqueness result, convergence study.

1 Introduction

We are interested here in the approximation of $\bar{u}$, solution to the so-called Stefan problem:

$$\partial_t \bar{u} - \Delta\zeta(\bar{u}) = f, \text{ in } \Omega \times (0,T) \tag{1}$$

with the following initial condition:

$$\bar{u}(\boldsymbol{x},0) = u_{\rm ini}(\boldsymbol{x}), \text{ for a.e. } x \in \Omega, \tag{2}$$

together with the homogeneous Dirichlet boundary condition:

$$\zeta(\bar{u}(\boldsymbol{x},t)) = 0 \text{ on } \partial\Omega \times (0,T), \tag{3}$$

under the following assumptions:

$$\Omega \text{ is an open bounded connected polyhedral subset of } \mathbb{R}^d,\ d \in \mathbb{N}^\star \text{ and } T > 0, \tag{4a}$$

$$u_{\rm ini} \in L^2(\Omega) \tag{4b}$$

$$f \in L^2(\Omega \times (0,T)), \tag{4c}$$

$$\zeta \in C^0(\mathbb{R}) \text{ is non–decreasing, Lipschitz continuous with Lipschitz constant } L_\zeta, \text{ and such that } \zeta(0) = 0, \tag{4d}$$

and

$$|\zeta(s)| \geq a|s| - b \text{ for all } s \in \mathbb{R} \text{ for some given values } a, b \in (0,+\infty). \tag{4e}$$

The Stefan Problem (1)-(2)-(3) arises in particular in the study of the heat equation in a nonmobile medium with two thermodynamical states, say solid and liquid. Denoting, for $(\boldsymbol{x},t) \in \Omega \times (0,T)$, by $\Theta(\boldsymbol{x},t)$ the temperature and by $X(\boldsymbol{x},t)$ the normalized mass of liquid phase per unit volume ($X(\boldsymbol{x},t) = 0$ means that the medium is solid at point $(\boldsymbol{x},t)$ and $X(\boldsymbol{x},t) = 1$ means that it is liquid), the internal energy $u(\boldsymbol{x},t)$ can be modeled by $\bar{u}(\boldsymbol{x},t) = H_c\Theta(\boldsymbol{x},t) + L_f X(\boldsymbol{x},t)$, where H_c denotes the heat capacity (assumed to be constant and identical for the liquid and the solid phases) and L_f denotes the latent heat of fusion at the given fusion temperature Θ_f. The heat equation can then be expressed by

$$\partial_t \bar{u} - \operatorname{div}(\lambda\nabla\Theta(\boldsymbol{x},t)) = f(\boldsymbol{x},t), \text{ in } \Omega \times (0,T), \tag{5}$$

where λ is the heat conductivity (assumed to be constant, isotropic and identical for the liquid and the solid phases). Dirichlet boundary conditions are assumed to be given on the temperature Θ, together with an initial condition given on $\bar{u}$. The thermodynamical equilibrium is then assumed to be expressed by

$$\begin{array}{ll} & (\Theta(\boldsymbol{x},t) \leq \Theta_f \text{ and } X(\boldsymbol{x},t) = 0) \\ \text{or} & (\Theta(\boldsymbol{x},t) = \Theta_f \text{ and } 0 \leq X(\boldsymbol{x},t) \leq 1) \\ \text{or} & (\Theta(\boldsymbol{x},t) \geq \Theta_f \text{ and } X(\boldsymbol{x},t) = 1) \qquad \text{a.e. } (\boldsymbol{x},t) \in \Omega \times (0,T). \end{array} \tag{6}$$

We may then remark that, under condition (6), $X(\boldsymbol{x},t)$ and $\Theta(\boldsymbol{x},t)$ can be expressed by $X(\boldsymbol{x},t) = \xi(\bar{u}(\boldsymbol{x},t))$ and $\Theta(\boldsymbol{x},t) = \frac{1}{\lambda}\zeta(\bar{u}(\boldsymbol{x},t))$, with

$$\xi(s) = \min(\max(\frac{s - H_c\Theta_f}{L_f}, 0), 1) \text{ and } \zeta(s) = \lambda\frac{s - L_f\xi(s)}{H_c}, \ \forall s \in \mathbb{R}.$$

Plugging the preceding expression of $\Theta(\boldsymbol{x},t)$ as function of $\bar{u}(\boldsymbol{x},t)$ in (5) leads to (1), in which the function ζ is Lipschitz continuous (it is in fact continuous and piecewise affine), nondecreasing, and constant on the interval $[H_c\Theta_f, H_c\Theta_f + L_f]$. Many results are known in this situation, in particular the fact that, if $f = 0$ and if the measure of the set $\{\boldsymbol{x} \in \Omega, \bar{u}(\boldsymbol{x},t) \in [H_c\Theta_f, H_c\Theta_f + L_f]\}$ (called the "mushy region") is equal to zero at $t = 0$, then it remains equal to zero for all $t > 0$, and a discontinuity of $\bar{u}$ between the values $H_c\Theta_f$ and $H_c\Theta_f + L_f$ may move inside the domain (see [5]). Therefore, Problem (1)-(2)-(3) has to be considered in a weak sense, which includes the Rankine-Hugoniot condition for the conservation of $\bar{u}$ in the case of discontinuities. A function $\bar{u}$ is said to be a weak solution of Problem (1)-(2)-(3) if the following holds:

$$\begin{aligned}
&\bar{u} \in L^2(\Omega \times (0,T)), \ \zeta(\bar{u}) \in L^2(0,T; H_0^1(\Omega)),\\
&\int_0^T \int_\Omega (-\bar{u}(\boldsymbol{x},t)\partial_t\varphi(\boldsymbol{x},t) + \nabla\zeta(\bar{u})(\boldsymbol{x},t) \cdot \nabla\varphi(\boldsymbol{x},t))\, \mathrm{d}\boldsymbol{x}\mathrm{d}t - \int_\Omega u_{\mathrm{ini}}(\boldsymbol{x})\varphi(\boldsymbol{x},0)\mathrm{d}\boldsymbol{x}\\
&\qquad = \int_0^T \int_\Omega f(\boldsymbol{x},t)\varphi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t, \ \forall\varphi \in C_c^\infty(\Omega \times [0,T[), \quad (7)
\end{aligned}$$

where we denote by $C_c^\infty(\Omega\times[0,T[)$ the set of the restrictions of functions of $C_c^\infty(\Omega\times]-\infty,T[)$ to $\Omega \times [0,T[$.

We also recall that, in mathematical finance, some derivative of the price of an American option is the solution of some Stefan problem [3, 4], whose computation is equivalent to the resolution of a variational inequality.

Let us first mention that the first proof of existence of a solution to Problem (7) has been provided in [1]. This proof relies on the convergence, as $\varepsilon > 0$ tends to zero, of the solution $\bar{u}_\varepsilon$ of the following strictly parabolic regularization of (1):

$$\partial_t\bar{u}_\varepsilon - \Delta(\zeta(\bar{u}_\varepsilon) + \varepsilon\bar{u}_\varepsilon) = f(\boldsymbol{x},t), \text{ in } \Omega \times (0,T). \quad (8)$$

It is easy to prove that $\partial_t\bar{u}_\varepsilon$ remains bounded in $L^2(0,T;H^{-1}(\Omega))$, and that $\zeta(\bar{u}_\varepsilon)$ remains bounded in $L^2(0,T;H_0^1(\Omega))$ for all $\varepsilon > 0$. But no compactness can be deduced from these two bounds. One remarkable idea in [1] is to show that $\|\zeta(\bar{u}_\varepsilon) - \zeta(\bar{u}_\varepsilon)(\cdot,\cdot+\tau)\|_{L^2(\Omega\times(0,T))}$ tends uniformly to 0 with τ. Then Kolmogorov's theorem allows to build a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ converging to 0 such that the sequence $(\zeta(\bar{u}_{\varepsilon_m}))_{m\in\mathbb{N}}$ converges to some function χ in $L^2(\Omega \times (0,T))$. A $L^\infty(0,T;L^2(\Omega))$ bound on $\bar{u}_\varepsilon$ allows to extract a subsequence from the sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ (identically denoted) such that there exists $\bar{u} \in L^\infty(0,T;L^2(\Omega))$ such that $(\bar{u}_{\varepsilon_m})_{m\in\mathbb{N}}$ weakly converges to $\bar{u}$. Then Minty's trick (which is available thanks to the monotonicity of ζ) provides that $\chi = \zeta(\bar{u})$ (this is detailed in Lemma B.1).

These ideas have been used in the study of the convergence of a finite volume method [13]. In this paper, $\Delta-$admissible meshes are used, in the sense that in each control volume K, there exists a given point $\boldsymbol{x}_K$ such that, for two neighboring control volumes K and L sharing the interface σ_{KL}, the line $(\boldsymbol{x}_K, \boldsymbol{x}_L)$ is parallel to the normal vector $\boldsymbol{n}_{K,L}$ to σ_{KL}, oriented from $\boldsymbol{x}_K$ to $\boldsymbol{x}_L$. Then the approximation of $\bar{u}$ in the control volume K (resp. L) is denoted by u_K (resp. u_L), and the approximation of $\nabla\zeta(\bar{u}) \cdot \boldsymbol{n}_{K,L}$ at this interface is given by the so-called two-point flux approximation $\frac{\zeta(u_L)-\zeta(u_K)}{\mathrm{d}(\boldsymbol{x}_K,\boldsymbol{x}_L)}$. A major advantage can be drawn from this approximation: multiplying the discrete scheme by u_K and summing on the control volumes leads to expressions such as $(\zeta(u_L) - \zeta(u_K))(u_L - u_K)$. Thanks to the Cauchy-Schwarz inequality, it is easy to prove that this expression is greater than $(\eta(u_L) - \eta(u_K))^2$, where η is a primitive of $(\zeta')^{1/2}$ (recall that a Lipschitz continuous function is absolutely continuous, and therefore a.e. derivable, and it is the primitive of its derivative, see e.g. [17, page 373]; moreover ζ' is bounded and therefore $(\zeta')^{1/2} \in L^1_{\mathrm{loc}}(\mathbb{R})$). Then functional properties (including a discrete maximum principle), similar to that of the continuous problem, can be shown, and the weak convergence of u and $\nabla\eta(u)$ in L^2, as well as the strong convergence of $\eta(u)$ in L^2 (implying that of $\zeta(u)$) thanks to monotony arguments and Kolmogorov's theorem, are proved. Stronger convergence properties (*i.e.* convergence of u and $\nabla\eta(u)$ in L^2) could then easily be shown, following the ideas developed in the present paper (see Lemma 3.2, Theorem 3.3 and Remark 1).

Unfortunately, if the diffusion term $\Delta\zeta(\bar{u})$ is approximated by a method other than the two-point flux approximation, say by a general finite element method or mixed finite element method, all the results obtained from the multiplication by u no longer hold.

Concerning the uniqueness of the solution of (7), several results are given in the literature under various hypotheses. A uniqueness theorem was proved in [13] with more restrictive assumptions on Ω than (4), and a uniqueness theorem for nonlinear convection-diffusion problems was proved in [6]. This uniqueness result was extended to the notion of entropy process solution in [12]; it allows to prove the convergence of a numerical scheme which extends the two-point flux approximation [13]. Therefore we have to check the uniqueness of the solution under the precise assumptions (4) made here. This is done in Theorem 4.1, the proof of which provides the opportunity to study the convergence of some gradient schemes to regular linear parabolic problems.

The purpose of this paper is to study the convergence of the so-called gradient schemes for the approximation of the Stefan problem given by its weak form (7). This framework includes, for example, the general case of the conforming finite elements (see [2], [9], [19] for the use of this specific method). It also includes the case of mixed finite element methods [11]. Gradient schemes have been studied in [14] for linear elliptic problems, and in [8] in the case of nonlinear Leray-Lions-type elliptic and parabolic problems. For such general methods, the monotony properties obtained from the two-point flux approximation cannot be used, and the multiplication by the solution u of the discrete scheme is of no use: in order to get estimates, one can only multiply by $\zeta(u)$. We have therefore introduced the additional hypothesis (4e) (which is not restrictive in practice, since it concerns the values of $\zeta(u)$ for large u, whereas u remains generally bounded) in order that a L^2 estimate on $\zeta(u)$ implies one on u.

This paper is organized as follows. We first apply the gradient discretization tools (described in an appendix) to the Stefan Problem in Section 2, and derive some estimates, which are used in Section 3 for the convergence analysis of gradient schemes. Then, turning to the uniqueness of the solution of (7), we first establish the existence of the continuous solution of a linear parabolic problem, showing the regularity which is further needed in the course of the uniqueness proof (see Section 4). Finally, numerical examples show the behavior of a particular gradient scheme, namely the VAG scheme [14] (see Section 5).

2 Approximation of the Stefan problem by space-time gradient discretizations

Let $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, (t^{(n)})_{n=0,\ldots,N})$ be a space-time discretization in the sense of Definition A.9 such that $\Pi_{\mathcal{D}}$ is a piecewise constant function reconstruction in the sense of Definition A.8. We define the following (implicit) scheme for the discretization of Problem (7). We consider a sequence $(u^{(n)})_{n=0,\ldots,N}$ such that:

$$\left\{\begin{array}{l} u^{(0)} \in X_{\mathcal{D},0}, \\ u^{(n+1)} \in X_{\mathcal{D},0},\ \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u = \Pi_{\mathcal{D}} \dfrac{u^{(n+1)} - u^{(n)}}{\delta t^{(n+\frac{1}{2})}}, \\ \displaystyle\int_\Omega \left(\delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\boldsymbol{x}) \Pi_{\mathcal{D}} v(\boldsymbol{x}) + \nabla_{\mathcal{D}} \zeta(u^{(n+1)})(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x} = \\ \dfrac{1}{\delta t^{(n+\frac{1}{2})}} \displaystyle\int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f(\boldsymbol{x},t) \Pi_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \mathrm{d}t, \quad \forall v \in X_{\mathcal{D},0},\ \forall n = 0, \ldots, N-1. \end{array}\right. \tag{9}$$

We again use the notations $\Pi_{\mathcal{D}}$ and $\nabla_{\mathcal{D}}$ for the definition of space-time dependent functions (note that we define these functions for all $t \in [0,T]$):

$$\begin{array}{l} \Pi_{\mathcal{D}} u(\boldsymbol{x}, 0) = \Pi_{\mathcal{D}} u^{(0)}(\boldsymbol{x}) \text{for a.e. } \boldsymbol{x} \in \Omega, \\ \Pi_{\mathcal{D}} u(\boldsymbol{x}, t) = \Pi_{\mathcal{D}} u^{(n+1)}(\boldsymbol{x}) \\ \Pi_{\mathcal{D}} \zeta(u)(\boldsymbol{x}, t) = \Pi_{\mathcal{D}} \zeta(u^{(n+1)})(\boldsymbol{x}) \\ \nabla_{\mathcal{D}} \zeta(u)(\boldsymbol{x}, t) = \nabla_{\mathcal{D}} \zeta(u^{(n+1)})(\boldsymbol{x}), \\ \text{for a.e. } \boldsymbol{x} \in \Omega,\ \forall t \in (t^{(n)}, t^{(n+1)}],\ \forall n = 0, \ldots, N-1. \end{array} \tag{10}$$

We also denote

$$\delta_{\mathcal{D}} u(\boldsymbol{x}, t) = \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\boldsymbol{x}),\ \text{for a.e. } (\boldsymbol{x}, t) \in \Omega \times (t^{(n)}, t^{(n+1)}),\ \forall n = 0, \ldots, N-1. \tag{11}$$

We finally introduce the primitive function

$$Z(s) = \int_0^s \zeta(x) \mathrm{d}x,\ \forall s \in \mathbb{R}. \tag{12}$$

which is used several times in the convergence proofs. We then have

$$Z(s) = \int_0^s \zeta(x) \mathrm{d}x = \int_0^s (\zeta(x) - \zeta(0)) \mathrm{d}x \le L_\zeta \int_0^s x \mathrm{d}x = L_\zeta \frac{s^2}{2},\ \forall s \in \mathbb{R}, \tag{13}$$

and, from Hypotheses (4d) and (4e),

$$Z(s) \geq \int_0^s \zeta(x)\frac{\zeta'(x)}{L_\zeta}\mathrm{d}x = \frac{\zeta(s)^2}{2L_\zeta} \geq \frac{a^2s^2 - 2b^2}{4L_\zeta},\ \forall s \in \mathbb{R}. \tag{14}$$

(where we have used the fact that ζ^2 is locally Lipschitz continuous, and therefore locally absolutely continuous).

Lemma 2.1 **($L^\infty(0,T;L^2(\Omega))$ estimate, discrete $L^2(0,T;H_0^1(\Omega))$ estimate and existence of a discrete solution)**

Under Hypotheses (4), let $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_\mathcal{D}, \nabla_\mathcal{D}, (t^{(n)})_{n=0,\ldots,N})$ be a space-time gradient discretization in the sense of Definition A.9 such that $\Pi_\mathcal{D}$ is a piecewise constant function reconstruction in the sense of Definition A.8. Then there exists at least one solution to Scheme (9), which satisfies

$$\begin{aligned}&\int_0^T \int_\Omega |\nabla_\mathcal{D}\zeta(u)(\boldsymbol{x},t)|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &+ \int_\Omega (Z(\Pi_\mathcal{D}u^{(N)}(\boldsymbol{x})) - Z(\Pi_\mathcal{D}u^0(\boldsymbol{x})))\mathrm{d}\boldsymbol{x} \leq \int_0^T \int_\Omega f(\boldsymbol{x},t)\Pi_\mathcal{D}\zeta(u)(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t.\end{aligned} \tag{15}$$

Moreover, let $C_P > 0$ such that $C_\mathcal{D} \leq C_P$, where $C_\mathcal{D}$ is the coercivity constant of the discretization (see Definition A.2 in the appendix) and let $C_{\text{ini}} > 0$ be such that $C_{\text{ini}} \geq \|u_{\text{ini}} - \Pi_\mathcal{D}u^{(0)}\|_{L^2(\Omega)}$; then there exists $C_1 > 0$, only depending on L_ζ, a, b, C_P, C_{ini} and f such that, for any solution u to this scheme,

$$\|\Pi_\mathcal{D}\zeta(u)\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1, \text{ and } \|\Pi_\mathcal{D}u\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1, \tag{16}$$

and

$$\|\nabla_\mathcal{D}\zeta(u)\|_{L^2(\Omega\times(0,T))^d} \leq C_1. \tag{17}$$

Proof

Before showing the existence of at least one discrete solution to Scheme (9), let us first prove if there exists a solution then it satisfies (15), (16) and (17). From the properties of function Z defined by (12), and using $\int_a^b \zeta(x)\mathrm{d}x = Z(b) - Z(a) = \zeta(b)(b-a) - \int_a^b \zeta'(x)(x-a)\mathrm{d}x$, we get, from Hypothesis (4d), that

$$\delta\!t^{(n+\frac{1}{2})}\delta_\mathcal{D}^{(n+\frac{1}{2})}u\ \Pi_\mathcal{D}\zeta(u^{(n+1)}) \geq \Pi_\mathcal{D}Z(u^{(n+1)}) - \Pi_\mathcal{D}Z(u^{(n)}). \tag{18}$$

We then let $v = \delta\!t^{(n+\frac{1}{2})}\zeta(u^{(n+1)})$ in (9), we sum the obtained equation for $n = 0,\ldots,m-1$ for a given $m = 1,\ldots,N$, and using (18), we get (15) replacing T by $t^{(m)}$ and $u^{(N)}$ by $u^{(m)}$. Thanks to the Cauchy-Schwarz inequality, we get that

$$\begin{aligned}&\|\Pi_\mathcal{D}Z(u^{(m)})\|_{L^1(\Omega)} + \int_0^{t^{(m)}} \|\nabla_\mathcal{D}\zeta(u)(\cdot,t)\|^2_{L^2(\Omega)^d}\mathrm{d}t \\ &\qquad \leq \|f\|_{L^2(\Omega\times(0,t^{(m)}))}\|\Pi_\mathcal{D}\zeta(u)\|_{L^2(\Omega\times(0,t^{(m)}))} + \|\Pi_\mathcal{D}Z(u^{(0)})\|_{L^1(\Omega)},\end{aligned}$$

which in turn yields, thanks to the Young inequality, and to (13) and (14),

$$\frac{1}{2L_\zeta}\|\Pi_\mathcal{D}\zeta(u^{(m)})\|^2_{L^2(\Omega)} + \int_0^{t^{(m)}} \|\nabla_\mathcal{D}\zeta(u)(\cdot,t)\|^2_{L^2(\Omega)^d}\mathrm{d}t \\ \le \frac{C_\mathcal{D}^2}{2}\|f\|^2_{L^2(\Omega\times(0,t^{(m)}))} + \frac{1}{2C_\mathcal{D}^2}\|\Pi_\mathcal{D}\zeta(u)\|^2_{L^2(\Omega\times(0,t^{(m)}))} + \frac{L_\zeta}{2}\|\Pi_\mathcal{D}u^{(0)}\|^2_{L^2(\Omega)}.$$

Using the definition (55) of $C_\mathcal{D}$, we prove the first estimate of (16) and the estimate (17). We get the second estimate of (16) by using the second part of (14).

The existence of a solution follows from these estimates by a now classical topological degree argument. Indeed, let $\theta \in [0,1]$, we introduce $\zeta_\theta(s) = \theta\zeta(s)+(1-\theta)as$, for any $s \in \mathbb{R}$. Replacing ζ by ζ_θ in the scheme, we get the same a priori estimates (16) and (17) independently of θ (remark that one can replace ζ by ζ_θ in (4d) and (4e), keeping the same values L_ζ, a and b since $a \le L_\zeta$). We conclude thanks to the Brouwer topological degree, since setting $\theta = 0$, we obtain the discretization of the heat equation, for which the existence of the solution is well–known. □

Lemma 2.2 **(Uniqueness results on the discrete solution)**

Under Hypotheses (4), let $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_\mathcal{D}, \nabla_\mathcal{D}, (t^{(n)})_{n=0,\dots,N})$ be a space-time gradient discretization in the sense of Definition A.9 such that $\Pi_\mathcal{D}$ is a piecewise constant function reconstruction in the sense of Definition A.8. Let $u^{(0)} \in X_{\mathcal{D},0}$ be given, and, for $n = 0,\dots,N-1$, let $u^{(n+1)} \in X_{\mathcal{D},0}$ be such that (9) holds. Then, for all $n = 0,\dots,N-1$, $\Pi_\mathcal{D}u^{(n+1)} \in L^2(\Omega)$ and $\zeta(u^{(n+1)}) \in X_{\mathcal{D},0}$ are unique.

Proof

Let us consider two solutions, denoted $u^{(n+1)}, \widetilde{u}^{(n+1)} \in X_{\mathcal{D},0}$, for some $n = 0,\dots,N-1$, such that (9) holds with $\Pi_\mathcal{D}u^{(n)}(\boldsymbol{x}) = \Pi_\mathcal{D}\widetilde{u}^{(n)}(\boldsymbol{x})$, for a.e. $\boldsymbol{x} \in \Omega$. We then subtract the corresponding equation with $\widetilde{u}^{(n+1)}$ to that with $u^{(n+1)}$. We get

$$\int_\Omega\Big(\frac{\Pi_\mathcal{D}(u^{(n+1)} - \widetilde{u}^{(n+1)})(\boldsymbol{x})}{\delta t^{(n+\frac12)}}\Pi_\mathcal{D}v(\boldsymbol{x}) + \\ \nabla_\mathcal{D}(\zeta(u^{(n+1)}) - \zeta(\widetilde{u}^{(n+1)}))(\boldsymbol{x})\cdot\nabla_\mathcal{D}v(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x} = 0, \forall v \in X_{\mathcal{D},0}. \quad (19)$$

We let $v = \zeta(u^{(n+1)}) - \zeta(\widetilde{u}^{(n+1)})$ in (19). Using Hypothesis (4d), we may write that

$$(\Pi_\mathcal{D}(u^{(n+1)} - \widetilde{u}^{(n+1)})(\boldsymbol{x}))\Pi_\mathcal{D}(\zeta(u^{(n+1)}) - \zeta(\widetilde{u}^{(n+1)}))(\boldsymbol{x}) \\ = (\Pi_\mathcal{D}u^{(n+1)}(\boldsymbol{x}) - \Pi_\mathcal{D}\widetilde{u}^{(n+1)}(\boldsymbol{x}))(\zeta(\Pi_\mathcal{D}u^{(n+1)}(\boldsymbol{x})) - \zeta(\Pi_\mathcal{D}\widetilde{u}^{(n+1)}(\boldsymbol{x}))) \ge 0,$$

which implies that

$$\int_\Omega |\nabla_\mathcal{D}(\zeta(u^{(n+1)}) - \zeta(\widetilde{u}^{(n+1)}))(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x} = 0,$$

and therefore that $\zeta(u^{(n+1)}) = \zeta(\widetilde{u}^{(n+1)})$. We then get, from (19), that

$$\int_\Omega \frac{\Pi_\mathcal{D}(u^{(n+1)} - \widetilde{u}^{(n+1)})(\boldsymbol{x})}{\delta t^{(n+\frac12)}}\Pi_\mathcal{D}v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0, \quad \forall v \in X_{\mathcal{D},0}.$$

It now suffices to let $v = u^{(n+1)} - \widetilde{u}^{(n+1)}$ in the preceding equation, to get that $\Pi_{\mathcal{D}} u^{(n+1)}(\boldsymbol{x}) = \Pi_{\mathcal{D}} \widetilde{u}^{(n+1)}(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Omega$.

□

In order to fulfill the hypotheses of discrete Alt–Luckhaus theorem B.3, let us study the time translates.

Lemma 2.3 (Estimate on the time translates)
Under Hypotheses (4), let $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, (t^{(n)})_{n=0,\ldots,N})$ be a space-time gradient discretization in the sense of Definition A.9 such that $\Pi_{\mathcal{D}}$ is a piecewise constant function reconstruction in the sense of Definition A.8. Then there exists $C_2 > 0$, only depending on L_ζ, a, b, $C_P > C_{\mathcal{D}}$, $C_{\rm ini} > \|u_{\rm ini} - \Pi_{\mathcal{D}} u^{(0)}\|_{L^2(\Omega)}$, f such that, for any solution u to Scheme (9),

$$\|\Pi_{\mathcal{D}}\zeta(u)(\cdot, \cdot + \tau) - \Pi_{\mathcal{D}}\zeta(u)(\cdot,\cdot)\|^2_{L^2(\Omega\times(0,T-\tau))} \le C_2(\tau + \delta t), \forall \tau \in (0,T). \tag{20}$$

Proof
In order to make the proof clear, let us give its principle, assuming that a solution $\bar{u}$ of the continuous equation (1) is regular enough. We write the time translate of this solution in $L^2(\Omega \times (0, T-\tau))$, for a step $\tau \in (0,T)$. We first note that

$$(\zeta(\bar{u}(\boldsymbol{x}, t+\tau)) - \zeta(\bar{u}(\boldsymbol{x}, t)))^2 \le L_\zeta(\zeta(\bar{u}(\boldsymbol{x}, t+\tau)) - \zeta(\bar{u}(\boldsymbol{x}, t)))(\bar{u}(\boldsymbol{x}, t+\tau) - \bar{u}(\boldsymbol{x}, t)),$$

which gives, using (1),

$$\begin{aligned}
&(\zeta(\bar{u}(\boldsymbol{x}, t+\tau)) - \zeta(\bar{u}(\boldsymbol{x}, t)))^2 \\
&\le L_\zeta(\zeta(\bar{u}(\boldsymbol{x}, t+\tau)) - \zeta(\bar{u}(\boldsymbol{x}, t))) \int_0^\tau \partial_t \bar{u}(\boldsymbol{x}, t+s) \mathrm{d}s \\
&\le L_\zeta(\zeta(\bar{u}(\boldsymbol{x}, t+\tau)) - \zeta(\bar{u}(\boldsymbol{x}, t))) \int_0^\tau (\Delta\zeta(\bar{u}(\boldsymbol{x}, t+s)) + f(\boldsymbol{x}, t+s)) \mathrm{d}s.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\int_0^{T-\tau} & \int_\Omega (\zeta(\bar{u}(\boldsymbol{x}, t+\tau)) - \zeta(\bar{u}(\boldsymbol{x}, t)))^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\le L_\zeta \int_0^\tau \int_0^{T-\tau} \int_\Omega (\zeta(\bar{u}(\boldsymbol{x}, t+\tau)) - \zeta(\bar{u}(\boldsymbol{x}, t))) \\
&\qquad\qquad (\Delta\zeta(\bar{u}(\boldsymbol{x}, t+s)) + f(\boldsymbol{x}, t+s)) \mathrm{d}\boldsymbol{x}\mathrm{d}t\mathrm{d}s \\
&\le L_\zeta \int_0^\tau \int_0^{T-\tau} \int_\Omega (-\nabla\zeta(\bar{u}(\boldsymbol{x}, t+\tau)) + \nabla\zeta(\bar{u}(\boldsymbol{x}, t))) \cdot \nabla\zeta(\bar{u}(\boldsymbol{x}, t+s)) \mathrm{d}\boldsymbol{x}\mathrm{d}t\mathrm{d}s \\
&\quad + L_\zeta \int_0^\tau \int_0^{T-\tau} \int_\Omega (\zeta(\bar{u}(\boldsymbol{x}, t+\tau)) - \zeta(\bar{u}(\boldsymbol{x}, t))) f(\boldsymbol{x}, t+s) \mathrm{d}\boldsymbol{x}\mathrm{d}t\mathrm{d}s.
\end{aligned}$$

Each product ab of the above right hand side is then bounded by $\frac{1}{2}(a^2+b^2)$, which allows to conclude, thanks to the continuous estimates similar to (16), that

$$\int_0^{T-\tau} \int_\Omega (\zeta(\bar{u}(\boldsymbol{x}, t+\tau)) - \zeta(\bar{u}(\boldsymbol{x}, t)))^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \le \tau C.$$

Let us now use the same ideas for the proof of (20). Let $\tau \in (0,T)$. Similarly using that L_ζ is a Lipschitz constant of ζ and ζ is nondecreasing, and using (59), the following inequality holds:

$$\int_{\Omega\times(0,T-\tau)} \Big(\Pi_{\mathcal D}\zeta(u)(\boldsymbol{x},t+\tau) - \Pi_{\mathcal D}\zeta(u)(\boldsymbol{x},t)\Big)^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \le L_\zeta \int_0^{T-\tau} A(t)\mathrm{d}t, \tag{21}$$

where, for almost every $t \in (0, T-\tau)$,

$$A(t) = \int_\Omega \Big(\Pi_{\mathcal D}\zeta(u)(\boldsymbol{x},t+\tau) - \Pi_{\mathcal D}\zeta(u)(\boldsymbol{x},t)\Big)\Big(\Pi_{\mathcal D}u(\boldsymbol{x},t+\tau) - \Pi_{\mathcal D}u(\boldsymbol{x},t)\Big)\mathrm{d}\boldsymbol{x}.$$

Let $t \in (0,T-\tau)$. Denoting $n_0(t)$, $n_1(t) = 0,\ldots,N-1$ such that $t^{(n_0(t))} \le t < t^{(n_0(t)+1)}$ and $t^{(n_1(t))} \le t+\tau < t^{(n_1(t)+1)}$, we may write

$$\begin{aligned} A(t) = \int_\Omega & \Big(\Pi_{\mathcal D}\zeta(u^{(n_1(t)+1)})(\boldsymbol{x}) - \Pi_{\mathcal D}\zeta(u^{(n_0(t)+1)})(\boldsymbol{x})\Big) \\ & \times \Big(\sum_{n=n_0(t)+1}^{n_1(t)} \delta t^{(n+\frac12)} \delta_{\mathcal D}^{(n+\frac12)} u(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}, \end{aligned}$$

which also reads

$$\begin{aligned} A(t) = \int_\Omega & \Big(\Pi_{\mathcal D}\zeta(u^{(n_1(t)+1)})(\boldsymbol{x}) - \Pi_{\mathcal D}\zeta(u^{(n_0(t)+1)})(\boldsymbol{x})\Big) \\ & \times \Big(\sum_{n=1}^{N-1} \chi_n(t,t+\tau)\delta t^{(n+\frac12)} \delta_{\mathcal D}^{(n+\frac12)} u(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{22}$$

with $\chi_n(t,t+\tau) = 1$ if $t^{(n)} \in (t,t+\tau]$ and $\chi_n(t,t+\tau) = 0$ if $t^{(n)} \notin (t,t+\tau]$. Letting $v = \zeta(u^{(n_1(t)+1)}) - \zeta(u^{(n_0(t)+1)})$ in Scheme (9), we get from (22)

$$\begin{aligned} A(t) = & \\ \sum_{n=1}^{N-1} & \chi_n(t,t+\tau) \\ & \times \int_\Omega \int_{t^{(n)}}^{t^{(n+1)}} f(\boldsymbol{x},t)\mathrm{d}t \Big(\Pi_{\mathcal D}\zeta(u^{(n_1(t)+1)})(\boldsymbol{x}) - \Pi_{\mathcal D}\zeta(u^{(n_0(t)+1)})(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x} \\ - \sum_{n=1}^{N-1} & \chi_n(t,t+\tau)\delta t^{(n+\frac12)} \\ & \times \int_\Omega \nabla_{\mathcal D}\zeta(u^{(n+1)})(\boldsymbol{x}) \cdot \Big(\nabla_{\mathcal D}\zeta(u^{(n_1(t)+1)})(\boldsymbol{x}) - \nabla_{\mathcal D}\zeta(u^{(n_0(t)+1)})(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Using the inequality $ab \le \frac12(a^2+b^2)$, this yields:

$$A(t) \le \frac12 A_0(t) + \frac12 A_1(t) + A_2(t) + A_3(t), \tag{23}$$

with

$$A_0(t) = \sum_{n=1}^{N-1} \chi_n(t,t+\tau)\delta t^{(n+\frac{1}{2})} \int_\Omega |\nabla_{\mathcal{D}}\zeta(u^{(n_0(t)+1)})(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x},$$

$$A_1(t) = \sum_{n=1}^{N-1} \chi_n(t,t+\tau)\delta t^{(n+\frac{1}{2})} \int_\Omega |\nabla_{\mathcal{D}}\zeta(u^{(n_1(t)+1)})(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x},$$

$$A_2(t) = \sum_{n=1}^{N-1} \chi_n(t,t+\tau)\delta t^{(n+\frac{1}{2})} \int_\Omega |\nabla_{\mathcal{D}}\zeta(u^{(n+1)})(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x},$$

and

$$A_3(t) = \sum_{n=1}^{N-1} \chi_n(t,t+\tau) \int_\Omega \int_{t^{(n)}}^{t^{(n+1)}} f(\boldsymbol{x},t)\mathrm{d}t\Big(\Pi_{\mathcal{D}}\zeta(u^{(n_1(t)+1)})(\boldsymbol{x}) - \Pi_{\mathcal{D}}\zeta(u^{(n_0(t)+1)})(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}.$$

Applying [16, Proposition 9.3] yields

$$\begin{aligned}
&\int_0^{T-\tau} A_0(t)\mathrm{d}t \le (\tau+\delta t)\|\nabla_{\mathcal{D}}\zeta(u)\|^2_{L^2(\Omega\times(0,T))} \\
&\text{and} \int_0^{T-\tau} A_1(t)\mathrm{d}t \le (\tau+\delta t)\|\nabla_{\mathcal{D}}\zeta(u)\|^2_{L^2(\Omega\times(0,T))},
\end{aligned} \tag{24}$$

as well as

$$\int_0^{T-\tau} A_2(t)\mathrm{d}t \le \tau\|\nabla_{\mathcal{D}}\zeta(u)\|^2_{L^2(\Omega\times(0,T))}, \tag{25}$$

and, with again the application of [16, Proposition 9.3], and using the Young inequality as well as (16), we obtain

$$\int_0^{T-\tau} A_3(t)\mathrm{d}t \le (\tau+\delta t)TC_1^2 + \tau\|f\|^2_{L^2(\Omega\times(0,T))}. \tag{26}$$

Using inequalities (21), (23), (24), (25) and (26), we conclude the proof of (20). □

3 Convergence results

Theorem 3.1

Let Hypotheses (4) be fulfilled. Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a consistent sequence of space-time gradient discretizations in the sense of Definition A.10, such that the associated sequence of approximate gradient approximations is limit–conforming (Definition A.4) and compact (Definition A.5, it is then coercive in the sense of Definition A.2), and such that, for all $m \in \mathbb{N}$, $\Pi_{\mathcal{D}_m}$ is a piecewise constant function reconstruction in the sense of Definition A.8. For any $m \in \mathbb{N}$, let u_m be a solution to Scheme (9), such that $\|u_{\rm ini} - \Pi_{\mathcal{D}_m} u_m^{(0)}\|_{L^2(\Omega)} \to 0$ as $m \to \infty$.

Then there exists $u \in L^2(\Omega \times (0,T))$ such that

1. $\Pi_{\mathcal{D}_m} u_m$ weakly converges in $L^2(\Omega \times (0,T))$ to u as $m \to \infty$,
2. $\Pi_{\mathcal{D}_m} \zeta(u_m)$ converges in $L^2(\Omega \times (0,T))$ to $\zeta(u)$ as $m \to \infty$,
3. $\zeta(u) \in L^2(0,T; H_0^1(\Omega))$ and $\nabla_{\mathcal{D}_m} \zeta(u_m)$ weakly converges in $L^2(\Omega \times (0,T))^d$ to $\nabla\zeta(u)$ as $m \to \infty$,

and u is the unique weak solution of Problem (7).

Proof

We consider, for all $m \in \mathbb{N}$, the spaces $B_m = \Pi_{\mathcal{D}_m} X_{\mathcal{D}_m,0} \subset L^2(\Omega)$, embedded with the norm

$$\|w\|_{B_m} = \inf\{\|u\|_{\mathcal{D}_m},\ \Pi_{\mathcal{D}_m} u = w\},\ \forall w \in B_m,\ \forall m \in \mathbb{N}.$$

The compactness hypothesis of $(\mathcal{D}_m)_{m\in\mathbb{N}}$ allows to enter into the framework of discrete Alt-Luckhaus' theorem B.3.

Thanks to Lemma 2.1, we get that Hypothesis (h1) of Theorem B.3 is satisfied. We classically identify $L^2(0,T;L^2(\Omega))$ and $L^2(\Omega \times (0,T))$, and we define, for $\tau \in (0,T)$, $g_m(\tau) = \|\Pi_{\mathcal{D}}\zeta(u)(\cdot,\cdot+\tau) - \Pi_{\mathcal{D}}\zeta(u)(\cdot,\cdot)\|_{L^2(\Omega\times(0,T-\tau))}$ and $g(\tau,t) = (C_2(\tau + \delta t))^{1/2}$. Thanks to Lemma 2.3 and to the continuity in means theorem (which implies that g_m is continuous in 0), we may apply Lemma B.2 and deduce that hypothesis (h2) of Theorem B.3 also holds. Therefore, there exists $\chi \in L^2(\Omega \times (0,T))$ such that $\Pi_{\mathcal{D}_m}\zeta(u_m)$ converges, up to the extraction of a subsequence, to χ in $L^2(\Omega \times (0,T))$. Again applying Lemma 2.1, we get that there exists $u \in L^2(\Omega \times (0,T))$ such that $\Pi_{\mathcal{D}_m} u_m$ weakly converges, up again to the extraction of a subsequence, to u in $L^2(\Omega \times (0,T))$. Thanks to Lemma B.1, we conclude that $\chi(\boldsymbol{x},t) = \zeta(u(\boldsymbol{x},t))$ for a.e. $(\boldsymbol{x},t) \in \Omega \times (0,T)$. It now remains to prove that u is the weak solution of Problem (7).

Let $m \in \mathbb{N}$, and let us denote $\mathcal{D} = \mathcal{D}_m$ (belonging to the above subsequence) and drop some indices m for the simplicity of the notation.

Let $\varphi \in C_c^\infty([0,T))$ and $w \in C_c^\infty(\Omega)$, and let $v \in X_{\mathcal{D},0}$ be such that

$$v = \operatorname*{argmin}_{z\in X_{\mathcal{D},0}} S_{\mathcal{D}}(w).$$

We take as test function v in (9) the function $\delta t^{(n+\frac{1}{2})}\varphi(t^{(n)})v$, and we sum the resulting equation on $n = 0, \ldots, N-1$. we get

$$T_1^{(m)} + T_2^{(m)} = T_3^{(m)}, \tag{27}$$

with

$$T_1^{(m)} = \sum_{n=0}^{N-1} \delta t^{(n+\frac{1}{2})}\varphi(t^{(n)}) \int_\Omega \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\boldsymbol{x})\Pi_{\mathcal{D}} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

$$T_2^{(m)} = \sum_{n=0}^{N-1} \delta t^{(n+\frac{1}{2})}\varphi(t^{(n)}) \int_\Omega \nabla_{\mathcal{D}}\zeta(u^{(n+1)})(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

and

$$T_3^{(m)} = \sum_{n=0}^{N-1} \varphi(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f(\boldsymbol{x},t)\Pi_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\mathrm{d}t.$$

Writing

$$T_1^{(m)} = -\int_0^T \varphi'(t) \int_\Omega \Pi_{\mathcal{D}} u(\boldsymbol{x},t)\Pi_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\mathrm{d}t - \varphi(0)\int_\Omega \Pi_{\mathcal{D}} u^{(0)}(\boldsymbol{x})\Pi_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x},$$

we get that

$$\lim_{m\to\infty} T_1^{(m)} = -\int_0^T \varphi'(t) \int_\Omega u(\boldsymbol{x},t) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\mathrm{d}t - \varphi(0)\int_\Omega u_{\mathrm{ini}}(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

We also immediately get that

$$\lim_{m\to\infty} T_2^{(m)} = \int_0^T \varphi(t) \int_\Omega \nabla\zeta(u)(\boldsymbol{x},t)\cdot\nabla w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\mathrm{d}t,$$

and

$$\lim_{m\to\infty} T_3^{(m)} = \int_0^T \varphi(t) \int_\Omega f(\boldsymbol{x},t) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\mathrm{d}t.$$

Since the set $\mathcal{T} = \{\sum_{i=1}^q \varphi_i(t) w_i(\boldsymbol{x}) \;:\; q\in\mathbb{N}, \varphi_i \in C_c^\infty[0,T), w_i\in C_c^\infty(\Omega)\}$ is dense in $C_c^\infty(\Omega\times[0,T))$, we conclude the proof of Theorem 3.1 thanks to the uniqueness of the limit solution proved in Theorem 4.1 below.

□

The next lemma states a continuous property, which is used below for proving that the convergence of $\nabla_{\mathcal{D}_m}\zeta(u_m)$ to $\nabla\zeta(u)$ is in fact strong.

Lemma 3.2

Under Hypotheses (4), let u be a solution of (7). Then the following property holds:

$$\begin{aligned}&\int_0^T \int_\Omega |\nabla\zeta(u)(\boldsymbol{x},t)|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_\Omega (Z(u(\boldsymbol{x},T)) - Z(u_{\mathrm{ini}}(\boldsymbol{x})))\mathrm{d}\boldsymbol{x}\\ &= \int_0^T \int_\Omega f(\boldsymbol{x},t)\zeta(u(\boldsymbol{x},t))\mathrm{d}\boldsymbol{x}\mathrm{d}t.\end{aligned} \tag{28}$$

Proof

We first notice that (7) implies that $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$ (and therefore $u\in C([0,T],H^{-1}(\Omega))$ with $u(0)=u_{\mathrm{ini}}$) and that we can write

$$\begin{aligned}&\int_0^T \left(\langle \partial_t u(t), w(t)\rangle + \int_\Omega \nabla\zeta(u)\cdot\nabla w \mathrm{d}\boldsymbol{x}\right)\mathrm{d}t\\ &= \int_0^T \int_\Omega f w \mathrm{d}\boldsymbol{x}\mathrm{d}t,\;\forall w\in L^2(0,T;H_0^1(\Omega)),\end{aligned} \tag{29}$$

denoting by $\langle\cdot,\cdot\rangle$ the duality product $(H^{-1}(\Omega), H_0^1(\Omega))$. We prolong u by $u(t)=u_{\mathrm{ini}}$ for all $t\le 0$, and by $u(t)=u(T)$ for all $t\ge T$.

Let $h \in (0,T)$. We consider $\alpha_h \in L^2(\mathbb{R}; H^{-1}(\Omega))$ defined by

$$\begin{aligned}\langle \alpha_h(t), w\rangle &= \frac{1}{h}\int_{t-h}^{t} \langle \partial_t u(s), w\rangle \mathrm{d}s \\ &= \int_\Omega \frac{1}{h}(u(\boldsymbol{x},t) - u(\boldsymbol{x},t-h))w(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \text{ for } t \in \mathbb{R}, \ \forall w \in H^1_0(\Omega).\end{aligned}$$

Then α_h tends to $\partial_t u$ in $L^2(\mathbb{R}; H^{-1}(\Omega))$ as $h \to 0$, which implies that

$$\begin{aligned}&\lim_{h\to 0}\int_{\mathbb{R}}\int_\Omega \frac{1}{h}(u(\boldsymbol{x},t) - u(\boldsymbol{x},t-h))w(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &+ \int_0^T\int_\Omega \nabla\zeta(u)\cdot\nabla w \mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_0^T\int_\Omega f w\mathrm{d}\boldsymbol{x}\mathrm{d}t, \ \forall w \in L^2(\mathbb{R}; H^1_0(\Omega)).\end{aligned}$$

Let us take $w = \zeta(u)$ in the above equation. We get

$$\begin{aligned}&\lim_{h\to 0}\int_{\mathbb{R}}\int_\Omega \frac{1}{h}(u(\boldsymbol{x},t) - u(\boldsymbol{x},t-h))\zeta(u(\boldsymbol{x},t))\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &+ \int_0^T\int_\Omega |\nabla\zeta(u)|^2\mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_0^T\int_\Omega f\ \zeta(u)\mathrm{d}\boldsymbol{x}\mathrm{d}t.\end{aligned}$$

Again observing that $\int_a^b \zeta(x)\mathrm{d}x = Z(b) - Z(a) = \zeta(b)(b-a) - \int_a^b \zeta'(x)(x-a)\mathrm{d}x$, which implies $Z(b) - Z(a) \le \zeta(b)(b-a)$, we get that

$$\begin{aligned}&\int_{\mathbb{R}}\int_\Omega \frac{1}{h}(u(\boldsymbol{x},t) - u(\boldsymbol{x},t-h))\zeta(u(\boldsymbol{x},t))\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &\ge \int_{\mathbb{R}}\int_\Omega \frac{1}{h}(Z(u(\boldsymbol{x},t)) - Z(u(\boldsymbol{x},t-h)))\mathrm{d}\boldsymbol{x}\mathrm{d}t.\end{aligned}$$

Since

$$\begin{aligned}&\int_{\mathbb{R}}\int_\Omega \frac{1}{h}(Z(u(\boldsymbol{x},t)) - Z(u(\boldsymbol{x},t-h)))\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &= \frac{1}{h}\int_T^{T+h}\int_\Omega Z(u(\boldsymbol{x},T))\mathrm{d}\boldsymbol{x}\mathrm{d}t - \frac{1}{h}\int_0^h\int_\Omega Z(u_{\mathrm{ini}}(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &= \int_\Omega (Z(u(\boldsymbol{x},T)) - Z(u_{\mathrm{ini}}(\boldsymbol{x})))\mathrm{d}\boldsymbol{x}.\end{aligned}$$

We may then pass to the limit $h \to 0$. We then obtain

$$\int_\Omega (Z(u(\boldsymbol{x},T)) - Z(u_{\mathrm{ini}}(\boldsymbol{x})))\mathrm{d}\boldsymbol{x} + \int_0^T\int_\Omega |\nabla\zeta(u)|^2\mathrm{d}\boldsymbol{x}\mathrm{d}t \le \int_0^T\int_\Omega f\ \zeta(u)\mathrm{d}\boldsymbol{x}\mathrm{d}t. \tag{30}$$

We then follow the same reasoning, defining $w = \zeta(u)$ and $\beta_h \in L^2(\mathbb{R}; H^{-1}(\Omega))$ by

$$\langle \beta_h(t), w\rangle = \frac{1}{h}\int_t^{t+h}\langle \partial_t u(s), w\rangle\mathrm{d}s, \text{ for } t \in \mathbb{R}, \ \forall w \in H^1_0(\Omega).$$

Remarking that $\int_a^b \zeta(x)\mathrm{d}x = Z(b) - Z(a) = \zeta(a)(b-a) + \int_a^b \zeta'(x)(b-x)\mathrm{d}x$, which implies $Z(b) - Z(a) \ge \zeta(a)(b-a)$, we get that

$$\begin{aligned}&\int_{\mathbb{R}}\int_\Omega \frac{1}{h}(u(\boldsymbol{x},t+h) - u(\boldsymbol{x},t))\zeta(u(\boldsymbol{x},t))\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &\le \int_{\mathbb{R}}\int_\Omega \frac{1}{h}(Z(u(\boldsymbol{x},t+h)) - Z(u(\boldsymbol{x},t)))\mathrm{d}\boldsymbol{x}\mathrm{d}t.\end{aligned}$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}}\int_\Omega \frac{1}{h}(Z(u(\boldsymbol{x},t+h))-Z(u(\boldsymbol{x},t)))\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&=\frac{1}{h}\int_{T-h}^{T}\int_\Omega Z(u(\boldsymbol{x},T))\mathrm{d}\boldsymbol{x}\mathrm{d}t-\frac{1}{h}\int_{-h}^{0}\int_\Omega Z(u_{\rm ini}(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&=\int_\Omega (Z(u(\boldsymbol{x},T))-Z(u_{\rm ini}(\boldsymbol{x})))\mathrm{d}\boldsymbol{x},
\end{aligned}$$

we may then pass to the limit $h\to 0$. We thus get

$$\int_\Omega (Z(u(\boldsymbol{x},T))-Z(u_{\rm ini}(\boldsymbol{x})))\mathrm{d}\boldsymbol{x}+\int_0^T\int_\Omega |\nabla\zeta(u)|^2\mathrm{d}\boldsymbol{x}\mathrm{d}t\ge\int_0^T\int_\Omega f\ \zeta(u)\mathrm{d}\boldsymbol{x}\mathrm{d}t,$$

which, in addition to (30), concludes the proof of (28).

□

We may now state the strong convergence of $\nabla_{\mathcal{D}}\zeta(u)$.

Theorem 3.3 (Strong convergence of the numerical scheme)
Under the same hypotheses as those of Theorem 3.1, Then there exists $u\in L^2(\Omega\times(0,T))\cap C([0,T],H^{-1}(\Omega))$ such that

1. $\Pi_{\mathcal{D}_m}u_m(t)$ weakly converges in $L^2(\Omega)$ to $u(t)$, for all $t\in[0,T]$, as $m\to\infty$,
2. $\Pi_{\mathcal{D}_m}\zeta(u_m)(t)$ converges in $L^2(\Omega)$ to $\zeta(u)(t)$, for all $t\in[0,T]$, as $m\to\infty$,
3. $\zeta(u)\in L^2(0,T;H^1_0(\Omega))$ and $\nabla_{\mathcal{D}_m}\zeta(u_m)$ converges in $L^2(\Omega\times(0,T))^d$ to $\nabla\zeta(u)$ as $m\to\infty$,

and u is the unique weak solution of Problem (7).

Proof
We first apply Theorem 3.1, which shows the weak convergence of $\Pi_{\mathcal{D}_m}u_m$ and $\nabla_{\mathcal{D}_m}\zeta(u_m)$, and the strong convergence of $\Pi_{\mathcal{D}_m}\zeta(u_m)$. Let $\varphi\in C^\infty_c(\Omega)$, and let w_m such that

$$w_m=\underset{z\in X_{\mathcal{D},0}}{\operatorname{argmin}}\,S_{\mathcal{D}}(\varphi).$$

We get from Scheme (9) and from the estimates given in Lemma 2.1 the following property: there exists C_3, only depending on the data introduced in 4, such that, for all $0\le s\le t$,

$$\left|\int_\Omega(\Pi_{\mathcal{D}_m}u_m(\boldsymbol{x},t)-\Pi_{\mathcal{D}_m}u_m(\boldsymbol{x},s))\Pi_{\mathcal{D}_m}w_m(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|\le(t-s+2\delta t_m)^{1/2}C_3\|w_m\|_{\mathcal{D}_m},$$

which gives, thanks to (16),

$$\begin{aligned}
&\left|\int_\Omega(\Pi_{\mathcal{D}_m}u_m(\boldsymbol{x},t)-\Pi_{\mathcal{D}_m}u_m(\boldsymbol{x},s))\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|\\
&\qquad\le(t-s+2\delta t_m)^{1/2}C_3\|w_m\|_{\mathcal{D}_m}+2C_1\|\varphi-\Pi_{\mathcal{D}_m}w_m\|_{L^2(\Omega)}.
\end{aligned}$$

Using $\sqrt{t-s+2\delta t_m} \le \sqrt{t-s} + \sqrt{2\delta t_m}$, we get

$$\left|\int_\Omega (\Pi_{\mathcal{D}_m} u_m(\boldsymbol{x},t) - \Pi_{\mathcal{D}_m} u_m(\boldsymbol{x},s))\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right| \le g(t-s, h_m^\varphi),$$

with $g(a,b) = \sqrt{a}C_4 + b$, with $C_4 = C_3 \max_m \|w_m\|_{\mathcal{D}_m}$ and $h_m^\varphi = (2\delta t_m)^{1/2} C_4 + 2C_1\|\varphi - \Pi_{\mathcal{D}_m} w_m\|_{L^2(\Omega)}$. We then may apply Theorem B.4 (given in the Appendix), proving that, for all $t \in [0,T]$, $\Pi_{\mathcal{D}_m} u_m(t)$ tends to $u(t)$ for the weak topology of $L^2(\Omega)$.

Thanks to the convexity of Z, we then get

$$\int_\Omega Z(u(\boldsymbol{x},T))\mathrm{d}\boldsymbol{x} \le \liminf_{m\to\infty} \int_\Omega Z(\Pi_{\mathcal{D}_m} u_m(\boldsymbol{x},T))\mathrm{d}\boldsymbol{x},$$

and we classically have, from the weak convergence property of $\nabla_{\mathcal{D}_m}\zeta(u_m)$,

$$\int_0^T \int_\Omega |\nabla\zeta(u)(\boldsymbol{x},t)|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \le \liminf_{m\to\infty} \int_0^T \int_\Omega |\nabla_{\mathcal{D}_m}\zeta(u_m)(\boldsymbol{x},t)|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t.$$

Therefore, we may pass to the limit sup as $m \to \infty$ in (15), and subtract (28). We thus obtain

$$\begin{aligned}&\limsup_{m\to\infty} \int_0^T \int_\Omega |\nabla_{\mathcal{D}_m}\zeta(u_m)(\boldsymbol{x},t)|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_\Omega Z(\Pi_{\mathcal{D}_m} u_m(\boldsymbol{x},T))\mathrm{d}\boldsymbol{x}\\ &\le \int_0^T \int_\Omega |\nabla\zeta(u)(\boldsymbol{x},t)|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_\Omega Z(u(\boldsymbol{x},T))\mathrm{d}\boldsymbol{x}.\end{aligned}$$

This shows that

$$\lim_{m\to\infty} \int_0^T \int_\Omega |\nabla_{\mathcal{D}_m}\zeta(u_m)(\boldsymbol{x},t)|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_0^T \int_\Omega |\nabla\zeta(u)(\boldsymbol{x},t)|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t,$$

which concludes the proof of the convergence of $\nabla_{\mathcal{D}_m}\zeta(u_m)$ to $\nabla\zeta(u)$ in $L^2(\Omega \times (0,T))^d$, and

$$\lim_{m\to\infty} \int_\Omega Z(\Pi_{\mathcal{D}_m} u_m(\boldsymbol{x},T))\mathrm{d}\boldsymbol{x} = \int_\Omega Z(u(\boldsymbol{x},T))\mathrm{d}\boldsymbol{x}. \tag{31}$$

Note that the preceding limit result holds in fact for all $t_0 \in [0,T]$ instead of T. We then remark that, thanks to the monotony of ζ, there holds [10, Lemma 2.3]

$$\frac{1}{2L_\zeta}(\zeta(a)-\zeta(b))^2 \le \int_a^b (\zeta(s)-\zeta(a))\mathrm{d}s = Z(b) - Z(a) - \zeta(a)(b-a),\ \forall a,b \in \mathbb{R}.$$

We then deduce

$$\begin{aligned}&\frac{1}{2L_\zeta}\int_\Omega (\zeta(\Pi_{\mathcal{D}_m} u_m(\boldsymbol{x},t_0)) - \zeta(u(\boldsymbol{x},t_0)))^2 \mathrm{d}\boldsymbol{x}\\ \le\ &\int_\Omega (Z(\Pi_{\mathcal{D}_m} u_m(\boldsymbol{x},t_0)) - Z(u(\boldsymbol{x},t_0)))\mathrm{d}\boldsymbol{x}\\ &-\textstyle\int_\Omega \zeta(u(\boldsymbol{x},t_0))(\Pi_{\mathcal{D}_m} u_m(\boldsymbol{x},t_0) - u(\boldsymbol{x},t_0))\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Since the right hand side tends to zero using (31) and the weak convergence of $\Pi_{\mathcal{D}_m} u_m(\cdot,t_0)$ to $u(\cdot,t_0)$, we conclude the convergence in $L^2(\Omega)$ of $\zeta(\Pi_{\mathcal{D}_m} u_m(\cdot,t_0))$ to $\zeta(u(\cdot,t_0))$, hence concluding the proof.

□

Remark 1 In the case where a two-point flux approximation is used instead of a gradient scheme, one can get with the same arguments that the approximation of u is strongly convergence at all times to the weak solution.

4 Proof of uniqueness by a regularized adjoint problem

Let us state and prove the uniqueness theorem, admitting some existence theorem proven below. The method is similar to that of [13], where the existence result for the adjoint problem is given under some regularity hypotheses on Ω which are not done in this paper.

THEOREM 4.1 Under Hypotheses (4), there exists at most one solution to (7).

Proof

Let u_1 and u_2 be two solutions of Problem (7). We set $u_d = u_1 - u_2$. Let us also define, for all $(\boldsymbol{x},t) \in \Omega \times \mathbb{R}_+^\star$, $q(\boldsymbol{x},t) = \dfrac{\zeta(u_1(\boldsymbol{x},t)) - \zeta(u_2(\boldsymbol{x},t))}{u_1(\boldsymbol{x},t) - u_2(\boldsymbol{x},t)}$ if $u_1(\boldsymbol{x},t) \neq u_2(\boldsymbol{x},t)$, else $q(\boldsymbol{x},t) = 0$. For all $T \in \mathbb{R}_+^\star$ and for all $\psi \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t\psi \in L^2(\Omega \times (0,T))$ and $\Delta\psi \in L^2(\Omega \times (0,T))$, we deduce from (7), approximating ψ by regular functions $\varphi \in C_c^\infty(\Omega \times [0,T))$, that

$$\int_0^T \int_\Omega u_d(\boldsymbol{x},t)\Big(\partial_t\psi(\boldsymbol{x},t) + q(\boldsymbol{x},t)\Delta\psi(\boldsymbol{x},t)\Big)\mathrm{d}\boldsymbol{x}\mathrm{d}t = 0. \tag{32}$$

Let $w \in C_c^\infty(\Omega \times (0,T))$. Let us denote, for $\varepsilon > 0$, $q_\varepsilon = q + \varepsilon$. We have

$$\varepsilon \leq q_\varepsilon(\boldsymbol{x},t) \leq L_\zeta + \varepsilon, \text{ for all } (\boldsymbol{x},t) \in \Omega \times (0,T),$$

and

$$\frac{(q_\varepsilon(\boldsymbol{x},t) - q(\boldsymbol{x},t))^2}{q_\varepsilon(\boldsymbol{x},t)} \leq \varepsilon. \tag{33}$$

Let ψ_ε be given by lemma 4.2 below, with $g = q_\varepsilon$. Substituting ψ by ψ_ε in (32) and using (37) give

$$|\int_0^T \int_\Omega u_d(\boldsymbol{x},t)w(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t| \leq |\int_0^T \int_\Omega u_d(\boldsymbol{x},t)(q_\varepsilon(\boldsymbol{x},t) - q(\boldsymbol{x},t))\Delta\psi_\varepsilon(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t|. \tag{34}$$

The Cauchy-Schwarz inequality, (38) and (33) imply

$$\begin{aligned}
&\Big[\int_0^T \int_\Omega u_d(\boldsymbol{x},t)(q_\varepsilon(\boldsymbol{x},t) - q(\boldsymbol{x},t))\Delta\psi_\varepsilon(\boldsymbol{x},t)|\mathrm{d}\boldsymbol{x}\mathrm{d}t\Big]^2 \\
&\leq \int_0^T \int_\Omega u_d(\boldsymbol{x},t)^2 \frac{(q(\boldsymbol{x},t) - q_\varepsilon(\boldsymbol{x},t))^2}{q_\varepsilon(\boldsymbol{x},t)}\mathrm{d}\boldsymbol{x}\mathrm{d}t \int_0^T \int_\Omega q_\varepsilon(\boldsymbol{x},t)\Big(\Delta\psi_\varepsilon(\boldsymbol{x},t)\Big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\leq \varepsilon \int_0^T \int_\Omega u_d(\boldsymbol{x},t)^2\mathrm{d}\boldsymbol{x}\mathrm{d}t \;\; 4T \int_0^T \int_\Omega |\nabla w(\boldsymbol{x},t)|^2\mathrm{d}\boldsymbol{x}\mathrm{d}t.
\end{aligned} \tag{35}$$

We deduce that the right hand side of (35) tends to zero as $\varepsilon \to 0$. Hence the left hand side of (34) also tends to zero as $\varepsilon \to 0$, which gives

$$|\int_0^T \int_\Omega u_d(\boldsymbol{x},t) w(\boldsymbol{x},t) \mathrm{d}\boldsymbol{x}\mathrm{d}t| = 0. \tag{36}$$

Since (36) holds for any function $w \in C_c^\infty(\Omega \times (0,T))$, we get that $u_d(\boldsymbol{x},t) = 0$ for a.e. $(\boldsymbol{x},t) \in \Omega \times (0,T)$, which concludes the proof of Theorem 4.1.

□

Let us now prove the properties of the function ψ, used in the course of the proof of Theorem 4.1.

Lemma 4.2

Under Hypothesis (4a), let $w \in L^2(0,T;H_0^1(\Omega))$ and $g \in L^\infty(\Omega \times (0,T))$ with $g(\boldsymbol{x},t) \in [g_{\min}, g_{\max}]$ with given $g_{\max} \geq g_{\min} > 0$ for a.e. $(\boldsymbol{x},t) \in \Omega \times (0,T)$. Then there exists at least one function ψ such that,

1. $\psi \in L^\infty(0,T;H_0^1(\Omega))$, $\partial_t \psi \in L^2(\Omega \times (0,T))$, $\Delta\psi \in L^2(\Omega \times (0,T))$ (hence $\psi \in C^0(0,T;L^2(\Omega))$),

2. $\psi(\cdot,T) = 0$,

3. the following holds

$$\partial_t \psi(\boldsymbol{x},t) + g(\boldsymbol{x},t)\Delta\psi(\boldsymbol{x},t) = w(\boldsymbol{x},t), \qquad \text{for a.e. } (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{37}$$

4. and

$$\int_0^T \int_\Omega g(\boldsymbol{x},t)\Big(\Delta\psi(\boldsymbol{x},t)\Big)^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \leq 4T \int_0^T \int_\Omega |\nabla w(\boldsymbol{x},t)|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t. \tag{38}$$

Proof

We first apply Lemma 4.4, which states the convergence of a gradient scheme to $\psi \in L^\infty(0,T;H_0^1(\Omega))$ with $\partial_t\psi \in L^2(\Omega \times (0,T))$ and $\Delta\psi \in L^2(\Omega \times (0,T))$ such that (37) holds, setting $\nu = 1/g$, $f = w/g$, $\mu(s) = s$, $\psi_{\rm ini} = 0$ and changing t in $-t$ (this ensures that Hypotheses (46) are fulfilled). Therefore the existence of ψ satisfying (37) follows. Let us prove that it satisfies (38). Approximating ψ by a sequence of regular functions and passing to the limit, we get that $\|\nabla\psi(\cdot)\|_{L^2(\Omega)^d} \in C^0([0,T])$ and that

$$\int_s^\tau \int_\Omega \partial_t\psi(\boldsymbol{x},t)\Delta\psi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t = -\frac{1}{2}\int_\Omega |\nabla\psi(\boldsymbol{x},\tau)|^2 \mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega |\nabla\psi(\boldsymbol{x},s)|^2 \mathrm{d}\boldsymbol{x},$$

for all $s < \tau \in [0,T]$ and

$$\int_s^\tau \int_\Omega w(\boldsymbol{x},t)\Delta\psi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t = -\int_s^\tau \int_\Omega \nabla w(\boldsymbol{x},t) \cdot \nabla\psi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t.$$

We thus obtain, multiplying (37) by $\Delta\psi(\boldsymbol{x},t)$ and integrating on $\Omega\times(0,\tau)$ for any $\tau\in[0,T]$,

$$\frac{1}{2}\int_\Omega |\nabla\psi(\boldsymbol{x},0)|^2\mathrm{d}\boldsymbol{x} \;-\; \frac{1}{2}\int_\Omega |\nabla\psi(\boldsymbol{x},\tau)|^2\mathrm{d}\boldsymbol{x} \;+\; \int_0^\tau\int_\Omega g(\boldsymbol{x},t)\Big(\Delta\psi(\boldsymbol{x},t)\Big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}t = \\ -\int_0^\tau\int_\Omega \nabla w(\boldsymbol{x},t)\cdot\nabla\psi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t. \tag{39}$$

Since $\nabla\psi(\cdot,T)=0$, letting $\tau=T$ in (39) leads to

$$\frac{1}{2}\int_\Omega |\nabla\psi(\boldsymbol{x},0)|^2\mathrm{d}\boldsymbol{x} + \int_0^T\int_\Omega g(\boldsymbol{x},t)\Big(\Delta\psi(\boldsymbol{x},t)\Big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}t = \\ -\int_0^T\int_\Omega \nabla w(\boldsymbol{x},t)\cdot\nabla\psi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t. \tag{40}$$

Integrating (39) with respect to $\tau\in(0,T)$ leads to

$$\begin{aligned}\frac{1}{2}\int_0^T\int_\Omega |\nabla\psi(\boldsymbol{x},\tau)|^2\mathrm{d}\boldsymbol{x}d\tau \le\; & \frac{T}{2}\int_\Omega |\nabla\psi(\boldsymbol{x},0)|^2\mathrm{d}\boldsymbol{x} \qquad + \\ & T\int_0^T\int_\Omega g(\boldsymbol{x},t)\Big(\Delta\psi(\boldsymbol{x},t)\Big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}t \qquad + \\ & T\int_0^T\int_\Omega |\nabla w(\boldsymbol{x},t)\cdot\nabla\psi(\boldsymbol{x},t)|\mathrm{d}\boldsymbol{x}\mathrm{d}t.\end{aligned} \tag{41}$$

Using (40) and (41), we get

$$\frac{1}{2}\int_0^T\int_\Omega |\nabla\psi(\boldsymbol{x},\tau)|^2\mathrm{d}\boldsymbol{x}d\tau \le 2T\int_0^T\int_\Omega |\nabla w(\boldsymbol{x},t)\cdot\nabla\psi(\boldsymbol{x},t)|\mathrm{d}\boldsymbol{x}\mathrm{d}t. \tag{42}$$

Thanks to the Cauchy-Schwarz inequality, the right hand side of (42) may be estimated as follows:

$$\Big[\int_0^T\int_\Omega |\nabla w(\boldsymbol{x},t)\cdot\nabla\psi(\boldsymbol{x},t)|\mathrm{d}\boldsymbol{x}\mathrm{d}t\Big]^2 \\ \le \int_0^T\int_\Omega |\nabla\psi(\boldsymbol{x},t)|^2\mathrm{d}\boldsymbol{x}\mathrm{d}t\int_0^T\int_\Omega |\nabla w(\boldsymbol{x},t)|^2\mathrm{d}\boldsymbol{x}\mathrm{d}t.$$

With (42), this implies

$$\Big[\int_0^T\int_\Omega |\nabla w(\boldsymbol{x},t)\cdot\nabla\psi(\boldsymbol{x},t)|\mathrm{d}\boldsymbol{x}\mathrm{d}t\Big]^2 \\ \le 4T\int_0^T\int_\Omega |\nabla w(\boldsymbol{x},t)\cdot\nabla\psi(\boldsymbol{x},t)|\mathrm{d}\boldsymbol{x}\mathrm{d}t\int_0^T\int_\Omega |\nabla w(\boldsymbol{x},t)|^2\mathrm{d}\boldsymbol{x}\mathrm{d}t.$$

Therefore,

$$\int_0^T\int_\Omega |\nabla w(\boldsymbol{x},t)\cdot\nabla\psi(\boldsymbol{x},t)|\mathrm{d}\boldsymbol{x}\mathrm{d}t \le 4T\int_0^T\int_\Omega |\nabla w(\boldsymbol{x},t)|^2\mathrm{d}\boldsymbol{x}\mathrm{d}t,$$

which, together with (40), yields (38).

□

In Lemma 4.2, we have used a result of existence of $\bar u \in L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega))$, such that $\Delta \bar u \in L^2(\Omega\times(0,T))$, solution to the following problem:

$$\nu(\boldsymbol{x},t)\partial_t \bar u(\boldsymbol{x},t) - \Delta \bar u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \text{ for a.e. } (\boldsymbol{x},t)\in\Omega\times(0,T) \tag{43}$$

with the following initial condition:

$$\bar u(\boldsymbol{x},0) = u_{\rm ini}(\boldsymbol{x}), \text{ for a.e. } x\in\Omega, \tag{44}$$

together with the homogeneous Dirichlet boundary condition:

$$\bar u(\boldsymbol{x},t) = 0 \text{ for a.e. } (\boldsymbol{x},t)\in\partial\Omega\times(0,T), \tag{45}$$

under the following assumptions (which are not exactly the standard ones done in the literature):

$$\Omega \text{ is an open bounded connected polyhedral subset of } \mathbb{R}^d,\ d\in\mathbb{N}^\star \text{ and } T>0, \tag{46a}$$

$$u_{\rm ini}\in H^1_0(\Omega) \tag{46b}$$

$$f\in L^2(\Omega\times(0,T)), \tag{46c}$$

and

$$\nu\in L^\infty(\Omega\times(0,T)) \text{ and } \nu(\boldsymbol{x},t)\in[\nu_{\min},\nu_{\max}] \text{ with given } \nu_{\max}\ge\nu_{\min}>0 \\ \text{for a.e. } (\boldsymbol{x},t)\in\Omega\times(0,T). \tag{46d}$$

This problem, issued from (32), is called the regularized adjoint problem to Problem (1). In order to prove the existence of a solution to Problem (43)-(44)-(45) under hypotheses (46), we consider an approximation of this solution, using a gradient scheme. Let $\mathcal{D} = (X_{\mathcal{D}},\Pi_{\mathcal{D}},\nabla_{\mathcal{D}},(t^{(n)})_{n=0,\dots,N})$ be a space-time discretization in the sense of Definition A.9. We define the fully implicit scheme for the discretization of Problem (52) by the sequence $(u^{(n)})_{n=0,\dots,N}\subset X_{\mathcal{D},0}$ such that:

$$\left\{\begin{array}{l} u^{(0)}\in X_{\mathcal{D},0},\\ u^{(n+1)}\in X_{\mathcal{D},0},\ \delta_{\mathcal{D}}^{(n+\frac12)}u = \Pi_{\mathcal{D}}\dfrac{u^{(n+1)}-u^{(n)}}{\delta t^{(n+\frac12)}},\\ \displaystyle\int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega \nu(\boldsymbol{x},t)\delta_{\mathcal{D}}^{(n+\frac12)}u(\boldsymbol{x})\Pi_{\mathcal{D}}v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}t\\ \displaystyle+\delta t^{(n+\frac12)}\int_\Omega \nabla_{\mathcal{D}}u^{(n+1)}(\boldsymbol{x})\cdot\nabla_{\mathcal{D}}v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega f(\boldsymbol{x},t)\Pi_{\mathcal{D}}v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}t,\\ \forall v\in X_{\mathcal{D},0},\ \forall n=0,\dots,N-1. \end{array}\right. \tag{47}$$

We then use the notations $\Pi_{\mathcal{D}}$ and $\nabla_{\mathcal{D}}$ for the definition of space-time dependent functions, defining

$$\begin{array}{l} \Pi_{\mathcal{D}}u(\boldsymbol{x},0) = \Pi_{\mathcal{D}}u^{(0)}(\boldsymbol{x}) \text{ and } \nabla_{\mathcal{D}}u(\boldsymbol{x},0) = \nabla_{\mathcal{D}}u^{(0)}(\boldsymbol{x}),\\ \Pi_{\mathcal{D}}u(\boldsymbol{x},t) = \Pi_{\mathcal{D}}u^{(n+1)}(\boldsymbol{x}) \text{ and } \nabla_{\mathcal{D}}u(\boldsymbol{x},t) = \nabla_{\mathcal{D}}u^{(n+1)}(\boldsymbol{x}),\\ \text{for a.e. } (\boldsymbol{x},t)\in\Omega\times(t^{(n)},t^{(n+1)}],\ \forall n=0,\dots,N-1. \end{array} \tag{48}$$

and

$$\delta_{\mathcal{D}} u(\boldsymbol{x}, t) = \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\boldsymbol{x}), \text{ for a.e. } (\boldsymbol{x}, t) \in \Omega \times (t^{(n)}, t^{(n+1)}), \ \forall n = 0, \ldots, N-1. \quad (49)$$

Let us state some estimates and the existence and uniqueness of the solution to the scheme.

Lemma 4.3 (Space-time estimates on $\delta_{\mathcal{D}} u$ and u.) Under Hypotheses (46), let $\mathcal{D}$ be a space-time gradient discretization in the sense of Definition A.9. Then, for any solution u to Scheme (47), we have:

$$\nu_{\min} \int_0^{t^{(m)}} \int_\Omega (\delta_{\mathcal{D}} u(\boldsymbol{x}, t))^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t + \|\nabla_{\mathcal{D}} u^m\|^2_{L^2(\Omega)^d} \\ \leq \|\nabla_{\mathcal{D}} u^{(0)}\|^2_{L^2(\Omega)^d} + \frac{1}{\nu_{\min}} \|f\|^2_{L^2(\Omega\times(0,T))}, \ \forall m = 1, \ldots, N. \quad (50)$$

As a result, there exists one and only one solution u to Scheme (47).

Proof
We set $v = u^{(n+1)} - u^{(n)}$ in (47) and we sum on $n = 0, \ldots, N-1$. We can then write

$$\frac{1}{2}|\nabla_{\mathcal{D}} u^{(n+1)}(\boldsymbol{x})|^2 - \frac{1}{2}|\nabla_{\mathcal{D}} u^{(n)}(\boldsymbol{x})|^2 \leq \nabla_{\mathcal{D}} u^{(n+1)}(\boldsymbol{x}) \cdot (\nabla_{\mathcal{D}} u^{(n+1)}(\boldsymbol{x}) - \nabla_{\mathcal{D}} u^{(n)}(\boldsymbol{x})).$$

Thanks to the Young inequality applied to the right hand side, we conclude (50), which ensures the existence and uniqueness of the solution to the linear Scheme (47), which leads to square linear systems.

□

We then have the following convergence lemma.

Lemma 4.4 (Convergence of the fully implicit scheme)
Let Hypotheses (46) be fulfilled. Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a consistent sequence of space-time gradient discretizations in the sense of Definition A.10, such that the associated sequence of approximate gradient approximations is consistent (Definition A.3), limit–conforming (Definition A.4) and compact (Definition A.5, it is then coercive in the sense of Definition A.2). For any $m \in \mathbb{N}$, let u_m be the solution to Scheme (47) for a given $u_m^{(0)} \in X_{\mathcal{D}_m,0}$, such that $\|\nabla u_{\text{ini}} - \nabla_{\mathcal{D}_m} u_m^{(0)})\|_{L^2(\Omega)^d} \to 0$ as $m \to \infty$.
Then there exist a sub-sequence of $(\mathcal{D}_m)_{m\in\mathbb{N}}$, again denoted $(\mathcal{D}_m)_{m\in\mathbb{N}}$, and a function $\bar{u} \in L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ such that

1. for all $t \in [0, T]$, $\Pi_{\mathcal{D}_m} u_m(t)$ converges in $L^2(\Omega)$ to $\bar{u}(t)$ with $\bar{u} \in L^\infty(0, T; L^2(\Omega))$ as $m \to \infty$,
2. $\delta_{\mathcal{D}_m} u_m$ weakly converges in $L^2(\Omega \times (0, T))$ to $\partial_t \bar{u}$ as $m \to \infty$,
3. $\nabla_{\mathcal{D}_m} u_m$ weakly converges in $L^\infty(0, T; L^2(\Omega)^d)$ to $\nabla \bar{u}$ as $m \to \infty$.
4. $\Delta \bar{u} \in L^2(\Omega \times (0, T))$,

5. (43)-(44)-(45) hold.

Proof

This proof has a few common points with that of [15, Lemma 4.4]. Thanks to (50), $\nabla_{\mathcal{D}_m} u_m$ remains bounded in $L^\infty(0,T;L^2(\Omega)^d)$ and $\Pi_{\mathcal{D}_m} u_m$ remains bounded in $L^2(\Omega\times(0,T))$. Since (50) also provides an $L^2(\Omega\times(0,T))$ estimate on $\delta_{\mathcal{D}_m} u_m$, this immediately provides an $L^\infty(0,T;L^2(\Omega))$ estimate on $(\Pi_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$, thanks to

$$\begin{aligned}\|\Pi_{\mathcal{D}} u^{(n)} - \Pi_{\mathcal{D}} u^{(p)}\|^2_{L^2(\Omega)} &= \|\sum_{k=p}^{n-1} \delta t^{(k+\frac12)} \delta_{\mathcal{D}}^{(k+\frac12)} u\|^2_{L^2(\Omega)} \\ &\le (t^{(n)} - t^{(p-1)}) \|\delta_{\mathcal{D}_m} u_m\|^2_{L^2(\Omega\times(0,T))}.\end{aligned}$$

Moreover, the above inequality, and the compactness hypothesis, allow to apply a variant of Ascoli's theorem similar to [15, Theorem 6.1], and whose proof is close to that of Theorem B.4. We deduce that there exists a function $\bar u \in C^0(0,T;L^2(\Omega))$ such that, up to the extraction of a subsequence, $(\Pi_{\mathcal{D}_m} u_m(t))_{m\in\mathbb{N}}$ converges to $\bar u(t)$ in $L^2(\Omega)$ for all $t\in[0,T]$. Using the limit-conformity of the discretization, we then get that $\bar u$ is such that

$$\begin{aligned}&\bar u \in L^\infty(0,T;H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega)),\\ &\bar u(\boldsymbol{x},0) = u_{\rm ini}(\boldsymbol{x}) \text{ for a.e. } \boldsymbol{x}\in\Omega,\end{aligned} \tag{51}$$

and using the consistency of the discretization in a similar way to the proof of Theorem 3.1, we get that

$$\int_0^T \int_\Omega (\nu\ \partial_t \bar u\ v + \nabla \bar u \cdot \nabla v)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_0^T \int_\Omega f\ v \mathrm{d}\boldsymbol{x}\mathrm{d}t, \forall v \in L^2(0,T;H^1_0(\Omega)). \tag{52}$$

Then (52) shows that $\Delta \bar u \in L^2(\Omega\times(0,T))$ and that (43)-(44)-(45) hold.

□

5 Numerical examples

5.1 The Vertex Approximate Gradient scheme

In the numerical tests proposed in this section, We use the Vertex Approximate Gradient scheme [14]. In this scheme, a primary mesh $\mathcal{M}$ in polyhedra is given. We assume that each element $K\in\mathcal{M}$ is strictly star-shaped with respect to some point $\boldsymbol{x}_K$. We denote by $\mathcal{E}_K$ the set of all interfaces $\overline{K}\cap\overline{L}$, for all neighbors of K denoted by $L\in\mathcal{M}$ and, for a boundary control volume, $\mathcal{E}_K$ also contains the element $\overline{K}\cap\partial\Omega$. Each $\sigma\in\mathcal{E}_K$ is assumed to be the reunion of $d-1$ simplices (segments if $d=2$, triangles if $d=3$) denoted $\tau\in\mathcal{S}_\sigma$. We denote by $\mathcal{V}_\sigma$ the set of all the vertices of σ, located at the boundary of σ, and by $\mathcal{V}^0_\sigma$ the set of all the internal vertices of σ. We assume that, for all $\boldsymbol{v}\in\mathcal{V}^0_\sigma$, there exists coefficients $(\alpha_{\boldsymbol{v}}^{\boldsymbol{x}})_{\boldsymbol{x}\in\mathcal{V}_\sigma}$, such that

$$\boldsymbol{v} = \sum_{\boldsymbol{x}\in\mathcal{V}_\sigma} \alpha_{\boldsymbol{v}}^{\boldsymbol{x}} \boldsymbol{x}, \text{ with } \sum_{\boldsymbol{x}\in\mathcal{V}_\sigma} \alpha_{\boldsymbol{v}}^{\boldsymbol{x}} = 1.$$

Therefore, the d vertices of any $\tau \in \mathcal{S}_\sigma$ are elements of $\mathcal{V}_\sigma^0 \cup \mathcal{V}_\sigma$. We denote by

$$\mathcal{V} = \bigcup_{\sigma\in\mathcal{E}} \mathcal{V}_\sigma,$$

and by $\mathcal{V}_K$ the set of all elements of $\mathcal{V}$ which are vertices of K. For any $K \in \mathcal{M}$, $\sigma \in \mathcal{E}_K$, $\tau \in \mathcal{S}_\sigma$, we denote by $S_{K,\tau}$ the d-simplex (triangle if $d = 2$, tetrahedron if $d = 3$) with vertex $\boldsymbol{x}_K$ and basis τ.

- We then define $X_\mathcal{D}$ as the set of all families $u = ((u_K)_{K\in\mathcal{M}}, (u_{\boldsymbol{v}})_{\boldsymbol{v}\in\mathcal{V}})$ and $X_{\mathcal{D},0}$ the set of all families $u \in X_\mathcal{D}$ such that $u_{\boldsymbol{v}} = 0$ for all $\boldsymbol{v} \in \mathcal{V} \cap \partial\Omega$.

- Disjoint arbitrary domains $V_{K,\boldsymbol{v}} \subset \bigcup_{\boldsymbol{v}\in\overline{\tau}} S_{K,\tau}$ are defined for all $\boldsymbol{v} \in \mathcal{V}_K$. Then the mapping $\Pi_\mathcal{D}$ is defined, for any $u \in X_\mathcal{D}$, by $\Pi_\mathcal{D} u(\boldsymbol{x}) = u_K$, for a.e. $\boldsymbol{x} \in K \setminus \bigcup_{\boldsymbol{v}\in\mathcal{V}_K} V_{K,\boldsymbol{v}}$, and $\Pi_\mathcal{D} u(\boldsymbol{x}) = u_{\boldsymbol{v}}$ for a.e. $\boldsymbol{x} \in V_{K,\boldsymbol{v}}$. It is important to notice that it is not in general necessary to provide a more precise geometric description of $V_{K,\boldsymbol{v}}$ than its measure.

- The mapping $\nabla_\mathcal{D}$ is defined, for any $u \in X_\mathcal{D}$, by $\nabla_\mathcal{D} u = \nabla\widehat{\Pi}_\mathcal{D} u$, where $\widehat{\Pi}_\mathcal{D} u$ is the continuous reconstruction which is affine in all $S_{K,\tau}$, for all $K \in \mathcal{M}$, $\sigma \in \mathcal{E}_K$ and $\tau \in \mathcal{S}_\sigma$, with the values u_K at $\boldsymbol{x}_K$, $u_{\boldsymbol{v}}$ at any vertex $\boldsymbol{v}$ of τ which belongs to $\mathcal{V}_\sigma$, and $\sum_{\boldsymbol{x}\in\mathcal{V}_\sigma} \alpha_{\boldsymbol{v}}^{\boldsymbol{x}} u_{\boldsymbol{x}}$ at any vertex $\boldsymbol{v}$ of τ which belongs to $\mathcal{V}_\sigma^0$.

The advantage of this scheme is that it allows to eliminate all values $(u_K)_{K\in\mathcal{M}}$ with respect to the values $(u_{\boldsymbol{v}})_{\boldsymbol{v}\in\mathcal{V}}$, leading to linear systems which are well suited to domain decomposition and parallel computing.

We then have the following result.

Lemma 5.1 (Gradient scheme properties of the VAG scheme)
We assume that, for all $m \in \mathbb{N}$, a gradient discretization $\mathcal{D}_m = (X_{\mathcal{D}_m}, \Pi_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m})$ is defined as specified in this section, respecting a uniform bound on the maximum value of the ratio between the diameter of all $K \in \mathcal{M}$ and that of the greatest ball with center $\boldsymbol{x}_K$ inscribed in K, and the ratio between the diameter of all $S_{K,\tau}$ and that of the greatest ball inscribed in $S_{K,\tau}$, for $K \in \mathcal{M}$, $\sigma \in \mathcal{E}_K$ and $\tau \in \mathcal{S}_\sigma$. We also assume that $h_{\mathcal{D}_m}$, the maximum diameter of all $K \in \mathcal{M}$, tends to 0 as $m \to \infty$. Then the sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is consistent, limit-conforming and compact (and therefore coercive).

Proof
For all $u \in X_\mathcal{D}$, the following property

$$\|\widehat{\Pi}_\mathcal{D} u - \Pi_\mathcal{D} u\|_{L^2(\Omega)} \le h_\mathcal{D} \|\nabla_\mathcal{D} u\|_{L^2(\Omega)^d}, \tag{53}$$

is resulting from $\widehat{\Pi}_\mathcal{D} u(\boldsymbol{x}) - \Pi_\mathcal{D} u(\boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{y}(\boldsymbol{x})) \cdot \nabla_\mathcal{D} u(\boldsymbol{x})$, for all $\boldsymbol{x} \in S_{K,\tau}$, where $\boldsymbol{y}(\boldsymbol{x}) \in S_{K,\tau}$ is the point of the mesh $\mathcal{M}$ defined by $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{x}_K$ if $\boldsymbol{x} \in S_{K,\tau} \setminus \bigcup_{\boldsymbol{v}\in\mathcal{V}_K} V_{K,\boldsymbol{v}}$, and by $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{v}$ if $\boldsymbol{x} \in S_{K,\tau} \cap V_{K,\boldsymbol{v}}$.

Let us check that the hypotheses of Lemma A.7 are satisfied, for some C only depending on regularity factors specified in the statement of the lemma, for $\widehat{\mathcal{D}}_m =$

$(X_{\mathcal{D}_m}, \widehat{\Pi}_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m})$. Then (58a) results from the interpolation results on the P^1 finite element under the regularity factor of the mesh, (58b) results from

$$\int_\Omega \Big(\nabla_{\mathcal{D}} u(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}) + \widehat{\Pi}_{\mathcal{D}} u(\boldsymbol{x}) \mathrm{div}\boldsymbol{\varphi}(\boldsymbol{x})\Big) \mathrm{d}\boldsymbol{x} = 0,$$

and (58c) results from

$$\|\widehat{\Pi}_{\mathcal{D}} u(\cdot + \boldsymbol{\xi}) - \widehat{\Pi}_{\mathcal{D}} u\|_{L^2(\mathbb{R}^d)} \le |\boldsymbol{\xi}| \|\nabla_{\mathcal{D}} u\|_{L^2(\Omega)^d}. \tag{54}$$

Therefore we obtain that the sequence $(\widehat{\mathcal{D}}_m)_{m\in\mathbb{N}}$ is consistent, limit-conforming and compact. From this result and thanks to (53), it is immediate to check that the sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is consistent and limit-conforming. We then remark that

$$\begin{aligned}\|\Pi_{\mathcal{D}} u(\cdot + \boldsymbol{\xi}) - \Pi_{\mathcal{D}} u\|_{L^2(\mathbb{R}^d)} \le\ & \|\Pi_{\mathcal{D}} u(\cdot + \boldsymbol{\xi}) - \widehat{\Pi}_{\mathcal{D}} u(\cdot + \boldsymbol{\xi})\|_{L^2(\mathbb{R}^d)} \\ & + \|\widehat{\Pi}_{\mathcal{D}} u(\cdot + \boldsymbol{\xi}) - \widehat{\Pi}_{\mathcal{D}} u\|_{L^2(\mathbb{R}^d)} + \|\widehat{\Pi}_{\mathcal{D}} u - \Pi_{\mathcal{D}} u\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

which leads, using (53) and (54), to

$$\|\Pi_{\mathcal{D}} u(\cdot + \boldsymbol{\xi}) - \Pi_{\mathcal{D}} u\|_{L^2(\mathbb{R}^d)} \le (2h_{\mathcal{D}} + |\boldsymbol{\xi}|) \|\nabla_{\mathcal{D}} u\|_{L^2(\Omega)^d}.$$

The application of (65) proved in Lemma B.2 leads to the relative compactness in B of any sequence $(\Pi_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$, if $u_m \in X_{\mathcal{D}_m,0}$ is such that $\|u_m\|_{\mathcal{D}_m}$ remains bounded. This completes the proof that the sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is compact.

□

5.2 A 2D test case on a variety of meshes

In this 2D test case, we approximate Stefan's problem (1) by using the VAG scheme previously described in the domain $\Omega = (0,1)^2$ with the following definition of $\zeta(\bar{u})$,

$$\zeta(\bar{u}) = \begin{cases} \bar{u} & \text{if } \bar{u} < 0, \\ \bar{u} - 1 & \text{if } \bar{u} > 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Dirichlet boundary condition is given by $\bar{u} = -1$ on $\partial\Omega$ and the initial condition (2) is given by $\bar{u}(\boldsymbol{x}, 0) = 2$. Four grids are used for the computations, a Cartesian grid with $32^2 = 1024$ cells, the same grid randomly perturbed, a triangular grids with 896 cells and a "Kershaw mesh" with 1089 cells as illustrated for example on the Figure 3 (such meshes are standard in the framework of underground engineering). The time simulation is 0.1 for a constant given time step of 0.001.

Figures 3, 4, 5 and 6 represent the discrete solution $u(\cdot, t)$ on all grids for $t = .025, 0.05, 0.075$ and 0.1. For a better comparison we have also plotted the interpolation of u along two lines of the mesh, the first line is horizontal and joins the two points $(0, 0.5)$ and $(1, 0.5)$, the second one is diagonal and joins points $(0, 0)$ and $(1, 1)$. The results thus obtained are shown in Figures 1 and 2.

We can see that the obtained results are weakly dependent on the grid, and that the interface between the regions $u < 0$ and $u > 1$ are located at the same place for all grids. It is worth to notice that this remains true for the very irregular Kershaw mesh, although it presents high ratios between the radii of inscribed balls and the diameter of some internal grid blocks.

A Appendix: gradient discretizations for diffusion problems

A gradient scheme can be viewed as a general framework for nonconforming approximation of elliptic or parabolic problems. These methods have been studied in [14] for linear elliptic problems, and in [8] in the case of nonlinear Leray-Lions-type elliptic and parabolic problems. The interest of the notion of gradient schemes is that it includes conforming finite elements with mass lumping (see Remark 6 below), mixed finite elements, hybrid mixed mimetic methods [7, 8], some discrete duality finite volume schemes, some particular Multi-point Flux Approximation and several other schemes. We begin with the discrete elements used for space partial differential equations.

Definition A.1 **(Gradient discretization)** A gradient discretization $\mathcal{D}$ for a space-dependent second order elliptic problem, with homogeneous Dirichlet boundary conditions, is defined by $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$, where:

1. the set of discrete unknowns $X_{\mathcal{D},0}$ is a finite dimensional vector space on $\mathbb{R}$,
2. the linear mapping $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)$ is the reconstruction of the approximate function,
3. the linear mapping $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)^d$ is the discrete gradient operator. It must be chosen such that $\| \cdot \|_{\mathcal{D}} := \|\nabla_{\mathcal{D}} \cdot \|_{L^2(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$.

Remark 2 (Boundary conditions.) The definition of $\|\cdot\|_{\mathcal{D}}$ depends on the considered boundary conditions. Here for simplicity we only consider homogeneous Dirichlet boundary conditions, but other conditions can easily be addressed. For example, in the case of homogeneous Neumann boundary conditions, we will use the notation $X_{\mathcal{D}}$ instead of $X_{\mathcal{D},0}$ for the discrete space, and define for example $\|u\|_{\mathcal{D}} := ((\int_\Omega \Pi_{\mathcal{D}} u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x})^2 + \|\nabla_{\mathcal{D}} u\|^2_{L^2(\Omega)^d})^{1/2}$; then Definition A.4 below has to be modified, changing $H_{\mathrm{div}}(\Omega)$ in $H_{\mathrm{div},0}(\Omega)$, the set of the elements of $H_{\mathrm{div}}(\Omega)$ with zero normal trace.

Definition A.2 (Coercivity) Let $\mathcal{D}$ be a gradient discretization in the sense of Definition A.1, and let $C_{\mathcal{D}}$ be the norm of the linear mapping $\Pi_{\mathcal{D}}$, defined by

$$C_{\mathcal{D}} = \max_{v \in X_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}} v\|_{L^2(\Omega)}}{\|v\|_{\mathcal{D}}}. \tag{55}$$

A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretizations is said to be **coercive** if there exists $C_P \in \mathbb{R}_+$ such that $C_{\mathcal{D}_m} \le C_P$ for all $m \in \mathbb{N}$.

Remark 3 (Discrete Poincaré inequality.)
In this case of homogeneous Dirichlet boundary conditions, (55) yields $\|\Pi_{\mathcal{D}} v\|_{L^2(\Omega)} \le C_{\mathcal{D}} \|\nabla_{\mathcal{D}} v\|_{L^2(\Omega)^d}$.

The consistency is ensured by a proper choice of the interpolation operator and discrete gradient.

DEFINITION A.3 (Consistency) Let $\mathcal{D}$ be a gradient discretization in the sense of Definition A.1, and let $S_{\mathcal{D}} : H_0^1(\Omega) \to [0, +\infty)$ be defined by

$$\forall \varphi \in H_0^1(\Omega)\,, \quad S_{\mathcal{D}}(\varphi) = \min_{v \in X_{\mathcal{D},0}} \left(\|\Pi_{\mathcal{D}} v - \varphi\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla \varphi\|_{L^2(\Omega)^d} \right). \tag{56}$$

A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretizations is said to be **consistent** if, for all $\varphi \in H_0^1(\Omega)$, $S_{\mathcal{D}_m}(\varphi)$ tends to 0 as $m \to \infty$.

Since we are dealing with nonconforming methods, we need that the dual of the discrete gradient be "close to" a discrete divergence.

DEFINITION A.4 (Limit-conformity) Let $\mathcal{D}$ be a gradient discretization in the sense of Definition A.1. We let $H_{\rm div}(\Omega) = \{\boldsymbol{\varphi} \in L^2(\Omega)^d, {\rm div}\boldsymbol{\varphi} \in L^2(\Omega)\}$ and $W_{\mathcal{D}}$: $H_{\rm div}(\Omega) \to [0, +\infty)$ be defined by

$$\forall \boldsymbol{\varphi} \in H_{\rm div}(\Omega)$$
$$W_{\mathcal{D}}(\boldsymbol{\varphi}) = \max_{u \in X_{\mathcal{D},0}\setminus\{0\}} \frac{1}{\|u\|_{\mathcal{D}}} \left| \int_\Omega \left(\nabla_{\mathcal{D}} u(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}) + \Pi_{\mathcal{D}} u(\boldsymbol{x}) {\rm div}\boldsymbol{\varphi}(\boldsymbol{x})\right) {\rm d}\boldsymbol{x} \right|. \tag{57}$$

A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretizations is said to be **limit-conforming** if, for all $\boldsymbol{\varphi} \in H_{\rm div}(\Omega)$, $W_{\mathcal{D}_m}(\boldsymbol{\varphi})$ tends to 0 as $m \to \infty$.

Dealing with generic non-linearity often requires compactness properties on the scheme.

DEFINITION A.5 (Compactness) A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretizations is said to be **compact** if, for all sequence $u_m \in X_{\mathcal{D}_m,0}$ such that $\|u_m\|_{\mathcal{D}_m}$ is bounded, the sequence $(\Pi_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ is relatively compact in $L^2(\Omega)$.

Let us state an important relation between compactness and coercivity.

LEMMA A.6 (COMPACTNESS IMPLIES COERCIVITY)
Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a compact sequence of gradient discretizations in the sense of Definition A.5. Then it is coercive in the sense of Definition A.2.

Proof
Let us assume that the sequence is not coercive. Then there exists a subsequence of $(\mathcal{D}_m)_{m\in\mathbb{N}}$ (identically denoted) such that, for all $m \in \mathbb{N}$, there exists $u_m \in X_{\mathcal{D}_m,0} \setminus \{0\}$ with

$$\lim_{m\to\infty} \frac{\|\Pi_{\mathcal{D}_m} u_m\|_{L^2(\Omega)}}{\|u_m\|_{\mathcal{D}_m}} = +\infty.$$

This means that, denoting by $v_m = u_m / \|u_m\|_{\mathcal{D}_m}$, $\lim_{m\to\infty} \|\Pi_{\mathcal{D}_m} v_m\|_{L^2(\Omega)} = +\infty$.
But we have $\|v_m\|_{\mathcal{D}_m} = 1$, and the compactness of the sequence of discretizations implies that the sequence $(\Pi_{\mathcal{D}_m} v_m)_{m\in\mathbb{N}}$ is relatively compact in $L^2(\Omega)$. This gives a contradiction.

□

Thanks to [14, Lemma 2.4], we may check the consistency and limit-conformity properties of given gradient schemes, only using dense subsets of the test functions spaces. The following lemma, useful in Section 5, is an immediate consequence of [14, Lemma 2.4] and of Kolmogorov's theorem.

Lemma A.7 (Sufficient conditions)

Let $\mathcal{F}$ be a family of gradient discretizations in the sense of Definition A.1. Assume that there exist $C, \nu \in (0,\infty)$ and, for all $\mathcal{D} \in \mathcal{F}$, a real value $h_{\mathcal{D}} \in (0,+\infty)$ such that:

$$S_{\mathcal{D}}(\varphi) \le C h_{\mathcal{D}} \|\varphi\|_{W^{2,\infty}(\Omega)}, \text{ for all } \varphi \in C_c^\infty(\Omega), \tag{58a}$$

$$W_{\mathcal{D}}(\boldsymbol{\varphi}) \le C h_{\mathcal{D}} \|\boldsymbol{\varphi}\|_{(W^{1,\infty}(\mathbb{R}^d))^d}, \text{ for all } \boldsymbol{\varphi} \in C_c^\infty(\mathbb{R}^d)^d, \tag{58b}$$

$$\max_{v \in X_{\mathcal{D},0}\setminus\{0\}} \frac{\|\Pi_{\mathcal{D}} v(\cdot + \boldsymbol{\xi}) - \Pi_{\mathcal{D}} v\|_{L^p(\mathbb{R}^d)}}{\|v\|_{\mathcal{D}}} \le C|\boldsymbol{\xi}|^\nu, \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d, \tag{58c}$$

where $C_{\mathcal{D}}, S_{\mathcal{D}}, W_{\mathcal{D}}$ are defined in this appendix.

Then, any sequence $(\mathcal{D}_m)_{m\in\mathbb{N}} \subset \mathcal{F}$ such that $h_{\mathcal{D}_m} \to 0$ as $m \to \infty$ is consistent, limit-conforming and compact (and therefore coercive).

Remark 4 In several cases, $h_{\mathcal{D}}$ stands for the mesh size: this is the case for the numerical schemes used in Section 5.

Definition A.8 (Piecewise constant function reconstruction)

Let $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ be a gradient discretization in the sense of Definition A.1, and I be the finite set of the degrees of freedom, such that $X_{\mathcal{D},0} = \mathbb{R}^I$. We say that $\Pi_{\mathcal{D}}$ is a piecewise constant function reconstruction if there exists a family of open subsets of Ω, denoted by $(\Omega_i)_{i\in I}$, such that $\bigcup_{i\in I} \overline{\Omega_i} = \overline{\Omega}$, $\Omega_i \cap \Omega_j = \emptyset$ for all $i \ne j$, and $\Pi_{\mathcal{D}} u = \sum_{i\in I} u_i \chi_{\Omega_i}$ for all $u = (u_i)_{i\in I} \in X_{\mathcal{D},0}$, where χ_{Ω_i} is the characteristic function of Ω_i.

Remark 5 Let us notice that $\|\Pi_{\mathcal{D}} \cdot\|_{L^2(\Omega)}$ is not requested to be a norm on $X_{\mathcal{D},0}$. Indeed, in several examples that can be considered, some degrees of freedom are involved in the reconstruction of the gradient of the function, but not in that of the function itself. Hence it can occur that some of the Ω_i are empty.

Remark 6 An important example of gradient discretization $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ in the sense of Definition A.1, such that $\Pi_{\mathcal{D}}$ is a piecewise constant function reconstruction in the sense of Definition A.8, is the case of the mass-lumping of conforming finite elements. Indeed, assuming that $(\xi_i)_{i\in I}$ is the basis of some finite–dimensional space $V_h \subset H_0^1(\Omega)$, we consider a family $(\Omega_i)_{i\in I}$, chosen such that

$$\|\sum_{i\in I} u_i \chi_{\Omega_i} - \sum_{i\in I} u_i \xi_i\|_{L^2(\Omega)} \le h \|\sum_{i\in I} u_i \nabla \xi_i\|_{L^2(\Omega)^d}, \ \forall u \in X_{\mathcal{D},0}.$$

We then define $\Pi_{\mathcal{D}}$ as in Definition A.8, and $\nabla_{\mathcal{D}} u = \sum_{i\in I} u_i \nabla \xi_i$. This is easily performed, considering P^1 conforming finite element, splitting each simplex in subsets defined by the highest barycentric coordinate, and defining Ω_i by the union of the subsets of the simplices connected to the vertex indexed by i.

Remark 7 Note that we have the two important following properties, in the case of a piecewise constant function reconstruction in the sense of Definition A.8:

$$g(\Pi_{\mathcal{D}} u(\boldsymbol{x})) = \Pi_{\mathcal{D}} g(u)(\boldsymbol{x}), \text{ for a.e. } \boldsymbol{x} \in \Omega, \ \forall u \in X_{\mathcal{D},0}, \ \forall g \in C(\mathbb{R}), \tag{59}$$

where for any continuous function $g \in C(\mathbb{R})$ and $u = (u_i)_{i\in I} \in X_{\mathcal{D},0}$, we classically denote by $g(u) = (g(u_i))_{i\in I} \in X_{\mathcal{D},0}$ and

$$\Pi_{\mathcal{D}} u(\boldsymbol{x}) \Pi_{\mathcal{D}} v(\boldsymbol{x}) = \Pi_{\mathcal{D}}(uv)(\boldsymbol{x}), \text{ for a.e. } \boldsymbol{x} \in \Omega, \ \forall u, v \in X_{\mathcal{D},0}, \tag{60}$$

where, for $u = (u_i)_{i\in I}$ and $v = (v_i)_{i\in I} \in X_{\mathcal{D},0}$, we denote by $uv = (u_i v_i)_{i\in I} \in X_{\mathcal{D},0}$.

Definition A.9 (Space-time gradient discretization) Under Hypothesis (4a), we say that $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, (t^{(n)})_{n=0,\ldots,N})$ is a space-time gradient discretization of $\Omega \times (0,T)$ if

- $(X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ is a gradient discretization of Ω, in the sense of Definition A.1,
- $t^{(0)} = 0 < t^{(1)} \ldots < t^{(N)} = T$.

We then set $\delta t^{(n+\frac{1}{2})} = t^{(n+1)} - t^{(n)}$, for $n = 0, \ldots, N-1$, and
$\delta t_{\mathcal{D}} = \max_{n=0,\ldots,N-1} \delta t^{(n+\frac{1}{2})}$.

Definition A.10 (Space-time consistency) A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of space-time gradient discretizations of $\Omega \times (0,T)$, in the sense of Definition A.9, is said to be **consistent** if it is consistent in the sense of Definition A.3 and if $\delta t_{\mathcal{D}_m}$ tends to 0 as $m \to \infty$.

B Appendix: technical results

The next result, which is known in the literature as the Minty trick, is used in the proof of the convergence theorem.

Lemma B.1 (Minty trick)
Let $\zeta \in C^0(\mathbb{R})$ be a nondecreasing function. Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 1$. Let $(u_n)_{n\in\mathbb{N}} \subset L^2(\Omega)$ such that
(i) there exists $u \in L^2(\Omega)$ such that $(u_n)_{n\in\mathbb{N}}$ weakly converges to u in $L^2(\Omega)$;
(ii) $(\zeta(u_n))_{n\in\mathbb{N}} \subset L^2(\Omega)$ and there exists $w \in L^2(\Omega)$ such that $(\zeta(u_n))_{n\in\mathbb{N}}$ weakly converges to w in $L^2(\Omega)$;
(iii) there holds:

$$\liminf_{n\to\infty} \int_\Omega u_n(\boldsymbol{x})\zeta(u_n(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} \leq \int_\Omega u(\boldsymbol{x})w(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{61}$$

Then $w(\boldsymbol{x}) = \zeta(u(\boldsymbol{x}))$, for a.e. $\boldsymbol{x} \in \Omega$.

Proof
We first consider, for any $v \in L^2(\Omega)$ such that $\zeta(v) \in L^2(\Omega)$,

$$A_n = \int_\Omega (\zeta(u_n(\boldsymbol{x})) - \zeta(v(\boldsymbol{x})))(u_n(\boldsymbol{x}) - v(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}.$$

Since ζ is a nondecreasing, we have $A_n \geq 0$. By weak/strong convergence and using (61), we get that

$$0 \leq \liminf_{n\to\infty} A_n \leq \int_\Omega (uw - u\zeta(v) - vw + v\zeta(v))\mathrm{d}\boldsymbol{x} = \int_\Omega (w - \zeta(v))(u - v)\mathrm{d}\boldsymbol{x}.$$

Hence we get that, for all $v \in L^2(\Omega)$ such that $\zeta(v) \in L^2(\Omega)$,

$$0 \le \int_\Omega (w - \zeta(v))(u - v)\mathrm{d}\boldsymbol{x} \le \int_\Omega (w - \zeta(0) + u - (\zeta(v) - \zeta(0) + v))(u - v)\mathrm{d}\boldsymbol{x}. \quad (62)$$

Since the mapping $\psi \;:\; s \to \zeta(s) - \zeta(0) + s$ is continuous from $\mathbb{R}$ to $\mathbb{R}$, strictly increasing and tends to infinity at infinity ($|\psi(s)| \ge |s|$ holds for all $s \in \mathbb{R}$), it is invertible. Let us denote by χ the reciprocal function to ψ, which therefore satisfies

$$|\chi(s)| \le |s| \text{ and } \zeta(\chi(s)) = s + \zeta(0) - \chi(s),\ \forall s \in \mathbb{R}. \quad (63)$$

We then get that, for all $z \in L^2(\Omega)$, the function $v = \chi(z)$ is such that $v \in L^2(\Omega)$ and $\zeta(v) \in L^2(\Omega)$. We then obtain from (62) that, denoting by $z_0 = w - \zeta(0) + u \in L^2(\Omega)$,

$$0 \le \int_\Omega (z_0 - z)(u - \chi(z))\mathrm{d}\boldsymbol{x},\ \forall z \in L^2(\Omega). \quad (64)$$

We then may take $z = z_0 - t\varphi$, with $t > 0$ and $\varphi \in C_c^\infty(\Omega)$ in (64). Dividing by $t > 0$, we obtain

$$\int_\Omega (u(\boldsymbol{x}) - \chi(z_0(\boldsymbol{x}) - t\varphi(\boldsymbol{x})))\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \ge 0.$$

Letting $t \to 0$ in the above equation, we get, by dominated convergence thanks to (63), that

$$\int_\Omega (u(\boldsymbol{x}) - \chi(z_0(\boldsymbol{x})))\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \ge 0.$$

Since the same inequality holds for $-\varphi$ instead of φ, we get

$$\int_\Omega (u(\boldsymbol{x}) - \chi(z_0(\boldsymbol{x})))\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0.$$

Since the above inequality holds for all $\varphi \in C_c^\infty(\Omega)$, we conclude that $u(\boldsymbol{x}) = \chi(z_0(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Omega$. This means that $\psi(u(\boldsymbol{x})) = z_0(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Omega$, which gives

$$\zeta(u(\boldsymbol{x})) - \zeta(0) + u(\boldsymbol{x}) = w(\boldsymbol{x}) - \zeta(0) + u(\boldsymbol{x}),\ \text{for a.e. } \boldsymbol{x} \in \Omega,$$

and the conclusion of the lemma follows.

□

The following result is used in the convergence proof, for proving the compactness of a particular scheme.

Lemma B.2 (Uniform limit.)

Let $N \in \mathbb{N}^\star$ and $(g_m)_{m\in\mathbb{N}}$ be a sequence of functions from $\mathbb{R}^N$ to $\mathbb{R}^+$ such that $g_m(0) = 0$ and g_m is continuous in 0. We assume that there exists a function $g \;:\; \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^+$, with $g(0,0) = 0$, continuous in $(0,0)$, and for all $m \in \mathbb{N}$, there exists $\mu_m \in \mathbb{R}^+$ verifying $\lim\limits_{m\to\infty} \mu_m = 0$, such that

$$g_m(\boldsymbol{\xi}) \le g(\boldsymbol{\xi}, \mu_m),\ \forall m \in \mathbb{N},\ \forall \boldsymbol{\xi} \in \mathbb{R}^N.$$

Then

$$\lim_{|\boldsymbol{\xi}|\to 0} \sup_{m\in\mathbb{N}} g_m(\boldsymbol{\xi}) = 0. \quad (65)$$

Proof

Let $\varepsilon > 0$. Let $\eta > 0$ be such that, for all $(\boldsymbol{\xi}, t) \in B(0,\eta) \times [0,\eta]$, $g(\boldsymbol{\xi}, t) \le \varepsilon$. Let $m_0 \in \mathbb{N}$ such that, for all $m > m_0$, $\mu_m \le \eta$. For all $m = 0, \ldots, m_0$, thanks to the continuity of g_m, there exists $\eta_m > 0$ such that, for all $\boldsymbol{\xi}$ verifying $|\boldsymbol{\xi}| \le \eta_m$, we have $g_m(\boldsymbol{\xi}) \le \varepsilon$.

We now take $\boldsymbol{\xi} \in \mathbb{R}^N$ such that $|\boldsymbol{\xi}| \le \min(\eta, (\eta_m)_{m=0,\ldots,m_0})$. We then get that, for all $m = 0, \ldots, m_0$, the inequality $g_m(\boldsymbol{\xi}) \le \varepsilon$ holds, and for all $m \in \mathbb{N}$ such that $m > m_0$, then $g(\boldsymbol{\xi}, \mu_m) \le \varepsilon$. Gathering the previous results gives (65).

□

We finally state a discrete version of Alt–Luckhaus theorem [1], whose proof is immediate following [18].

Theorem B.3 (Discrete Alt–Luckhaus theorem) Let $T > 0$, let B be a Banach space, and let $p \in [1, +\infty)$. Let $(B_m)_{m\in\mathbb{N}}$ be a sequence of normed subspaces of B such that, for any sequence $(w_m)_{m\in\mathbb{N}}$ such that $w_m \in B_m$ and $(\|w_m\|_{B_m})_{m\in\mathbb{N}}$ is bounded, then the set $\{w_m, m \in \mathbb{N}\}$ is relatively compact in B. Let $(v_m)_{m\in\mathbb{N}}$ such that $v_m \in L^p(0,T;B_m)$ for all $m \in \mathbb{N}$. We assume that

(h1) the sequence $(\|v_m\|_{L^p(0,T;B_m)})_{m\in\mathbb{N}}$ is bounded,

(h2) $\|v_m(\cdot + h) - v_m\|_{L^p(0,T-h;B)}$ tends to 0 as $h \in (0,T)$ tends to 0, uniformly with respect to $m \in \mathbb{N}$.

Then the set $\{v_m, m \in \mathbb{N}\}$ is relatively compact in $L^p(0,T;B)$.

Proof

Our aim is to apply Theorem 2.1 of [18]. We then prolong v_m by 0 on $(-\infty, 0) \cup (T, +\infty)$, for all $m \in \mathbb{N}$. Let us prove that $\|v_m(\cdot + h) - v_m\|_{L^p(\mathbb{R};B)}$ tends to 0 as $h \in (0,T)$ tends to 0, uniformly with respect to $m \in \mathbb{N}$. Let us first remark that there exists $C_N > 0$ such that,

$$\forall m \in \mathbb{N},\ \forall v \in B_m,\ \|v\|_B \le C_N \|v\|_{B_m}.$$

Indeed, otherwise one could, up to a subsequence of $(B_m)_{m\in\mathbb{N}}$, construct a sequence such that $\|v_m\|_{B_m} = 1$ and $\|v_m\|_B$ tends to infinity, which is in contradiction with the relative compactness in B of $\{v_m, m \in \mathbb{N}\}$. Hence we can define

$$C_B = \sup_{m\in\mathbb{N}} \|v_m\|^p_{L^p(0,T;B)}.$$

We have, for all $h \in (0,T)$,

$$\begin{aligned}\|v_m(\cdot + h) - v_m\|^p_{L^p(\mathbb{R};B)} = \quad & \|v_m(\cdot + h) - v_m\|^p_{L^p(0,T-h;B)} + \|v_m\|^p_{L^p(0,h;B)} \\ & + \|v_m\|^p_{L^p(T-h,T;B)}.\end{aligned}$$

Let us prove that

$$\lim_{h\to 0} \sup_{m\in\mathbb{N}} \|v_m\|^p_{L^p(0,h;B)} = 0. \tag{66}$$

Let $\varepsilon > 0$. We first choose $h_0 \in (0,T)$ such that, for all $h \in (0,h_0)$,

$$\|v_m(\cdot + h) - v_m\|^p_{L^p(0,T-h;B)} \le \varepsilon,\ \forall m \in \mathbb{N}. \tag{67}$$

Let $\tau \in (0, T-h_0)$, $h \in (0, h_0)$ and $m \in \mathbb{N}$ be given. We have

$$\int_0^\tau \|v_m(t)\|_B^p \mathrm{d}t \le 2^{p-1}\left(\int_0^\tau \|v_m(t+h)-v_m(t)\|_B^p \mathrm{d}t + \int_0^\tau \|v_m(t+h)\|_B^p \mathrm{d}t\right).$$

Thanks to (67), the above inequality gives

$$\int_0^\tau \|v_m(t)\|_B^p \mathrm{d}t \le 2^{p-1}\left(\varepsilon + \int_0^\tau \|v_m(t+h)\|_B^p \mathrm{d}t\right). \tag{68}$$

We then remark that

$$\int_0^{h_0}\int_0^\tau \|v_m(t+h)\|_B^p \mathrm{d}t\mathrm{d}h = \int_0^\tau\int_0^{h_0} \|v_m(t+h)\|_B^p \mathrm{d}h\mathrm{d}t$$
$$\le \int_0^\tau\int_0^T \|v_m(h)\|_B^p \mathrm{d}h\mathrm{d}t \le C_B\tau.$$

This proves that

$$h_0 \inf_{h\in(0,h_0)} \int_0^\tau \|v_m(t+h)\|_B^p \mathrm{d}t \le C_B\tau.$$

Taking the infimum on h in (68), we get, for all $\tau \in (0, T-h_0)$ and $m \in \mathbb{N}$,

$$\int_0^\tau \|v_m(t)\|_B^p \mathrm{d}t \le 2^{p-1}\left(\varepsilon + \frac{C_B\tau}{h_0}\right).$$

It now suffices to take $\tau \in (0, \min(T-h_0, \frac{h_0\varepsilon}{C_B}))$ for getting

$$\int_0^\tau \|v_m(t)\|_B^p \mathrm{d}t \le 2^p\varepsilon,\ \forall m \in \mathbb{N}.$$

This concludes the proof of (66). A similar proof can be done for proving that

$$\lim_{h\to 0}\sup_{m\in\mathbb{N}} \|v_m\|_{L^p(T-h,T;B)}^p = 0.$$

We thus conclude that

$$\lim_{h\to 0}\sup_{m\in\mathbb{N}} \|v_m(\cdot+h)-v_m\|_{L^p(\mathbb{R};B)}^p = 0,$$

which enables to apply Theorem 2.1 of [18], hence providing the conclusion of the proof.

□

Theorem B.4 Let Ω be an open bounded subset of $\mathbb{R}^2$, $a < b \in \mathbb{R}$ and $(u_m)_{m\in\mathbb{N}}$ be a sequence of functions from $[a,b]$ to $L^2(\Omega)$, such that there exists $C_1 > 0$ with

$$\|u_m(t)\|_{L^2(\Omega)} \le C_1,\ \forall m \in \mathbb{N},\ \forall t \in [a,b]. \tag{69}$$

We also assume that there exists a dense subset R of $L^2(\Omega)$ such that, for all $\varphi \in R$, there exists a function $g_\varphi\ :\ \mathbb{R}^+ \times \mathbb{R}^+$ with $g(0,0) = 0$, continuous in $(0,0)$ and a sequence $(h_m^\varphi)_{m\in\mathbb{N}}$ with $h_m^\varphi \ge 0$ and $\lim_{m\to\infty} h_m^\varphi = 0$ and such that

$$\left|\langle u_m(t_2)-u_m(t_1),\varphi\rangle_{L^2(\Omega),L^2(\Omega)}\right| \le g_\varphi(t_2-t_1, h_m^\varphi),\ \forall m \in \mathbb{N},\ \forall a \le t_1 \le t_2 \le b. \tag{70}$$

Then there exists $u \in L^\infty(a,b;L^2(\Omega))$ with $u \in C_w([a,b],L^2(\Omega))$ (where we denote by $C_w([a,b],L^2(\Omega))$ the set of functions from $[a,b]$ to $L^2(\Omega)$, continuous for the weak topology of $L^2(\Omega)$) and a subsequence of $(u_m)_{m\in\mathbb{N}}$, again denoted $(u_m)_{m\in\mathbb{N}}$, such that, for all $t \in [a,b]$, $u_m(t)$ converges to $u(t)$ for the weak topology of $L^2(\Omega)$.

Proof

The proof follows that of Ascoli's theorem. Let $(t_p)_{p\in\mathbb{N}}$ be a sequence of real numbers, dense in $[a,b]$. Due to (69), for each $p \in \mathbb{N}$, we may extract from $(u_m(t_p))_{m\in\mathbb{N}}$ a subsequence which is convergent to some element of $L^2(\Omega)$ for the weak topology of $L^2(\Omega)$. Using a diagonal method, we can choose a sub-sequence, again denoted $(u_m)_{m\in\mathbb{N}}$, such that $(u_m(t_p))_{m\in\mathbb{N}}$ s is weakly convergent for all $p \in \mathbb{N}$. For any $t \in [a,b]$ and $v \in L^2(\Omega)$, we then prove that the sequence $(\langle u_m(t),v\rangle_{L^2(\Omega),L^2(\Omega)})_{m\in\mathbb{N}}$ is a Cauchy sequence. Indeed, let $\varepsilon > 0$ be given. We first choose $\varphi \in R$ such that $\|\varphi - v\|_{L^2(\Omega)} \le \varepsilon$. Let $\eta > 0$ such that, for all $(s,t) \in [0,\eta]^2$, we have $g_\varphi(s,t) \le \varepsilon$. Then, we choose $p \in \mathbb{N}$ such that $|t - t_p| \le \eta$. Since $(\langle u_m(t_p),\varphi\rangle_{L^2(\Omega),L^2(\Omega)})_{m\in\mathbb{N}}$ is a Cauchy sequence, we choose $n_0 \in \mathbb{N}$ such that, for $k,l \ge n_0$,

$$\left|\langle u_k(t_p) - u_l(t_p),\varphi\rangle_{L^2(\Omega),L^2(\Omega)}\right| \le \varepsilon,$$

and such that $h_k^\varphi, h_l^\varphi \le \eta$. We then get, using (70),

$$\left|\langle u_k(t) - u_l(t),\varphi\rangle_{L^2(\Omega),L^2(\Omega)}\right| \le g_\varphi(|t-t_p|,h_k^\varphi) + g_\varphi(|t-t_p|,h_l^\varphi) + \varepsilon,$$

which gives

$$\left|\langle u_k(t) - u_l(t),\varphi\rangle_{L^2(\Omega),L^2(\Omega)}\right| \le 3\varepsilon.$$

This proves that the sequence $(\langle u_m(t),v\rangle_{L^2(\Omega),L^2(\Omega)})_{m\in\mathbb{N}}$ converges. Since

$$|\langle u_m(t),v\rangle_{L^2(\Omega),L^2(\Omega)}| \le C_1\|v\|_{L^2(\Omega)},$$

we get the existence of $u(t) \in L^2(\Omega)$ such that $(u_m(t))_{m\in\mathbb{N}}$ converges to $u(t)$ for the weak topology of $L^2(\Omega)$. Then $u \in C_w([a,b],L^2(\Omega))$ is obtained by passing to the limit $m \to \infty$ in (70), and by using the density of R in $L^2(\Omega)$.

□

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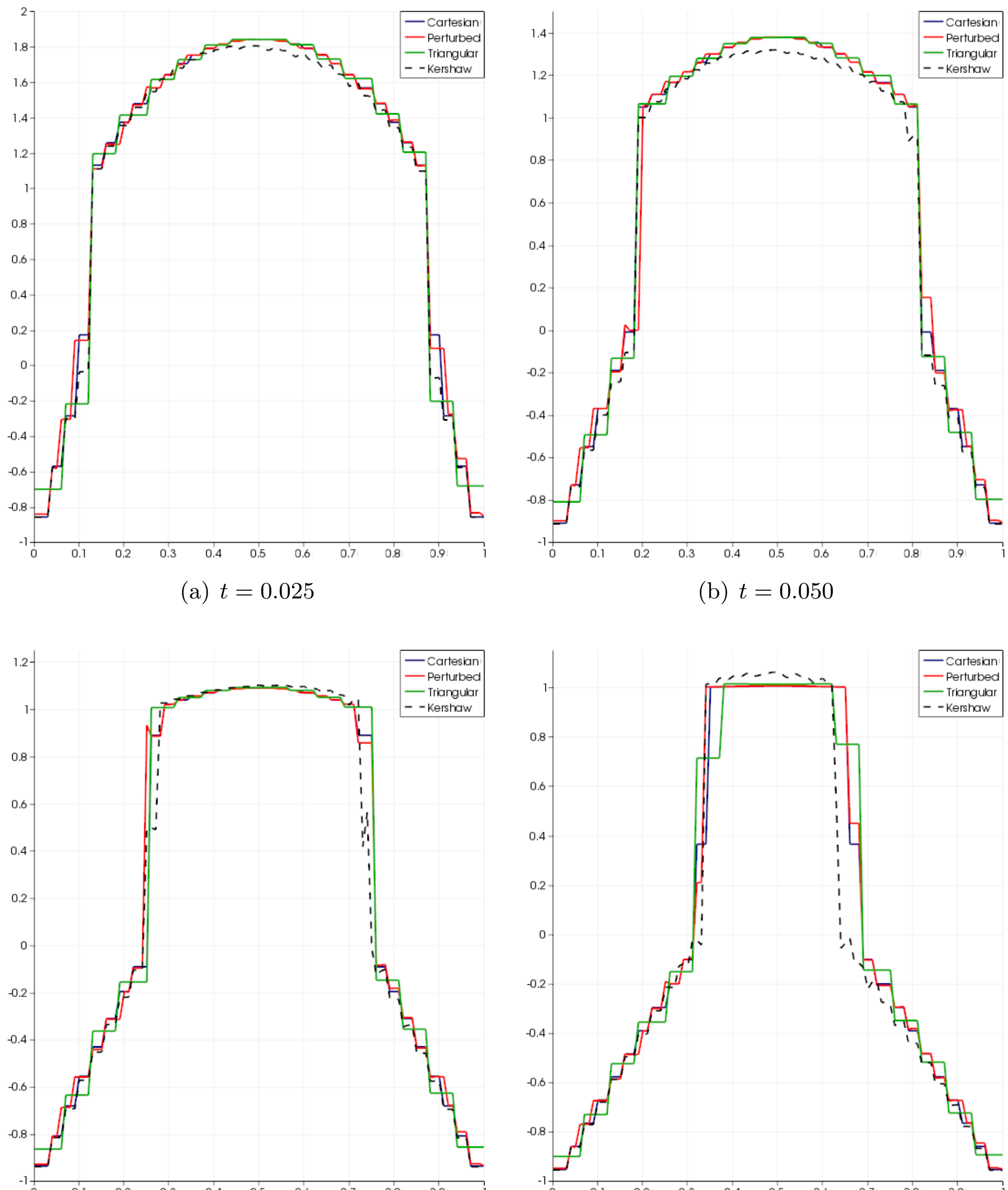


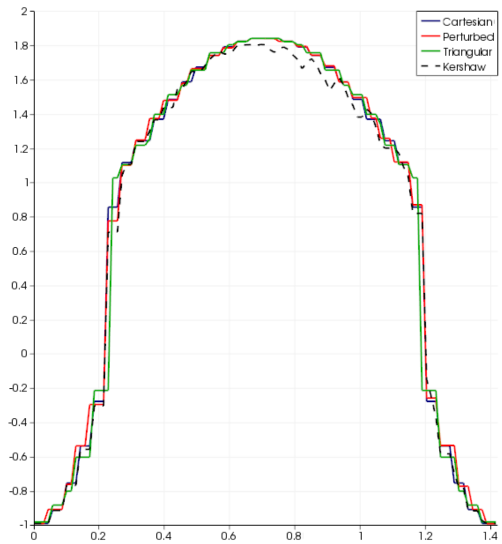
(a) $t = 0.025$

(b) $t = 0.050$

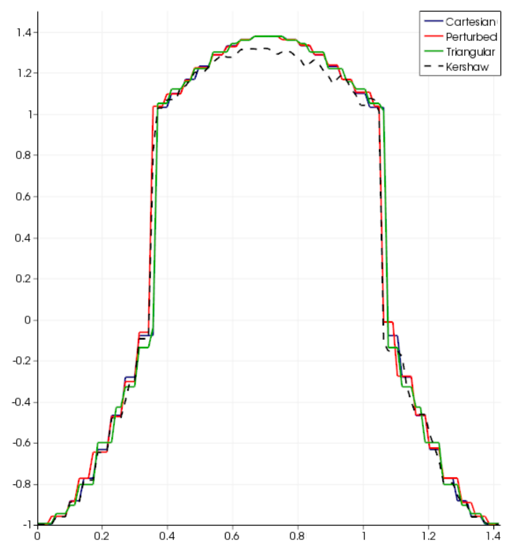
(c) $t = 0.075$

(d) $t = 0.1$

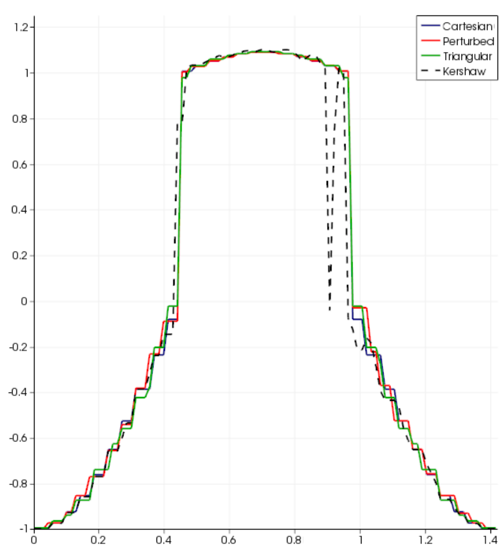
Figure 1: Interpolation of u along the line $[\boldsymbol{x}, 0.5]$ of the mesh for each grids : the Cartesian in blue, the perturbed in red, the triangular in green and the Kershaw in black dashed.



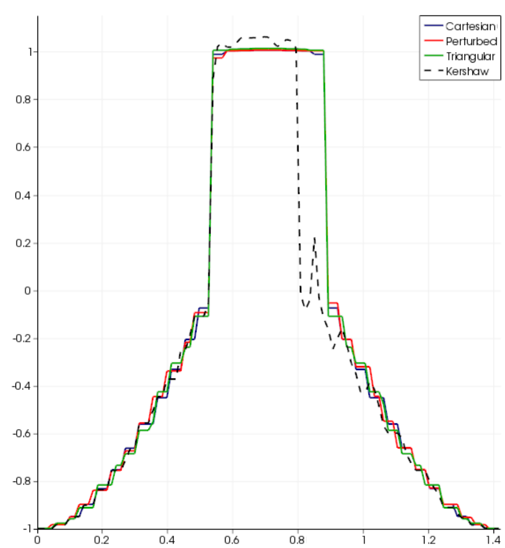
(a) $t = 0.025$



(b) $t = 0.050$

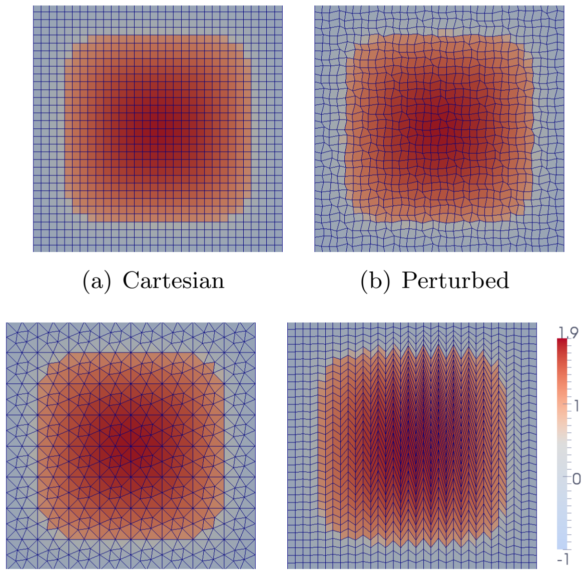


(c) $t = 0.075$



(d) $t = 0.1$

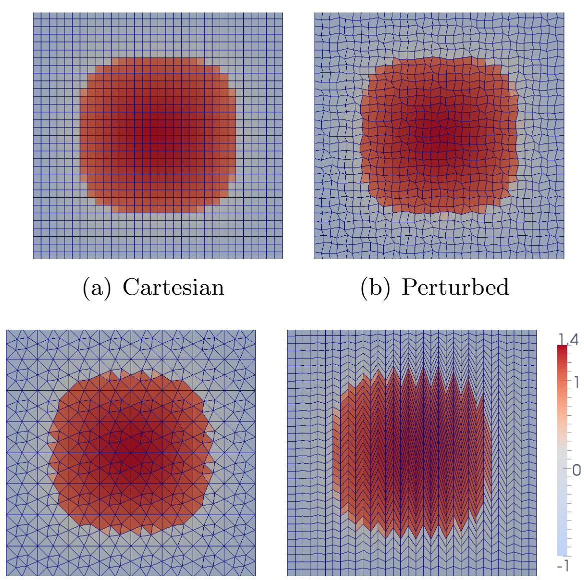
Figure 2: Interpolation of u along a diagonal axe of the mesh for each grids : the Cartesian in blue, the perturbed in red, the triangular in green and the Kershaw in black dashed.



(a) Cartesian (b) Perturbed

(c) Triangular (d) Kershaw

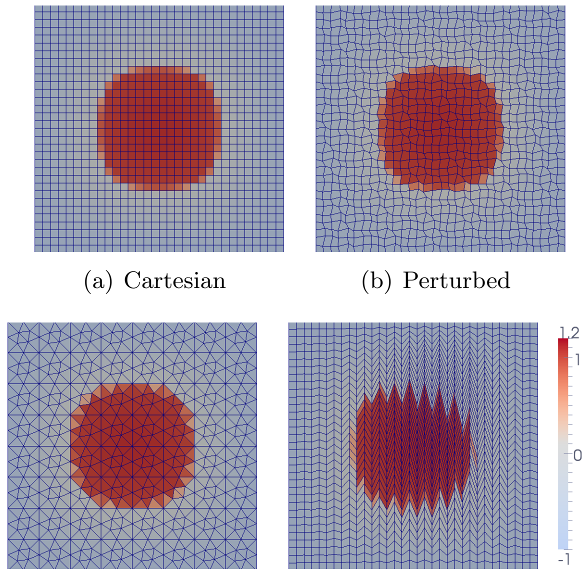
Figure 3: Discrete solution u on all grids at $t = 0.025$.



(a) Cartesian (b) Perturbed

(c) Triangular (d) Kershaw

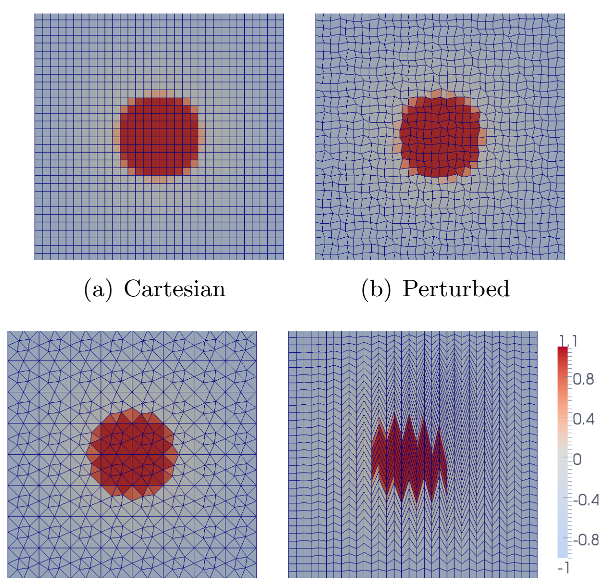
Figure 4: Discrete solution u on all grids at $t = 0.050$.



(a) Cartesian (b) Perturbed

(c) Triangular (d) Kershaw

Figure 5: Discrete solution u on all grids at $t = 0.075$.



(a) Cartesian (b) Perturbed

(c) Triangular (d) Kershaw

Figure 6: Discrete solution u on all grids at $t = 0.1$.