Kinetic energy control in explicit Finite Volume discretizations of the incompressible and compressible Navier-Stokes equations

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Abstract

We prove that, under a **cfl** condition, the explicit upwind finite volume discretization of the convection operator $\mathcal{C}(u) = \partial_t(\rho u) + \mathrm{div}(u\boldsymbol{q})$, with a given density ρ and momentum $\boldsymbol{q}$, satisfies a discrete kinetic energy decrease property, provided that the convection operator satisfies a "consistency-with-the-mass-balance property", which can be simply stated by saying that it vanishes for a constant advected field u.

Key words : Compressible Navier-Stokes equations, Finite Volume discretizations, Stability, Kinetic Energy.

1 Introduction

Let ρ and $\boldsymbol{q}$ be a given scalar and a given vector smooth function respectively, defined over a domain Ω of $\mathbb{R}^d$, $d = 2$ or $d = 3$, and such that the following identity holds in Ω:

$$\partial_t \rho + \mathrm{div}\boldsymbol{q} = 0. \tag{1}$$

Let u be a smooth scalar function defined over Ω. If $\boldsymbol{q}$ vanishes on the boundary, the following stability identity is known to hold:

$$\int_\Omega \big[\partial_t(\rho u) + \mathrm{div}(u\boldsymbol{q})\big]\, u \,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\,\frac{d}{dt}\int_\Omega \rho u^2 \,\mathrm{d}\boldsymbol{x}. \tag{2}$$

When ρ stands for the density and $\boldsymbol{q}$ for the momentum, equation (1) is the usual mass balance in variable density flows. Choosing for u a component of the velocity, equation (2) yields the central argument of the kinetic energy conservation theorem.

A discrete analogue of this result has been proven in [2] for an implicit discretization of the convection operator for u, *i.e.* $\mathcal{C}(u) = \partial_t(\rho u) + \mathrm{div}(u\boldsymbol{q})$, and is a central argument of the stability of schemes for low Mach number flows [1], barotropic monophasic [2] or diphasic [3] compressible flows. The aim of the present short note is to prove that the same stability result holds for an explicit upwind discretization of $\mathcal{C}(u)$, under a **cfl** condition. This result yields the (conditional) stability of the semi-implicit version (*i.e.* with an explicit convection term in the momentum balance) of the discretizations for incompressible or compressible barotropic Navier-Stokes equations studied in [1, 2, 3], provided that the other terms in these schemes remain unchanged: the viscous term, discretized in an implicit way, and the pressure gradient term, with a discrete gradient built as the transposed of the discrete divergence to allow a control of the pressure work. Such a semi-implicit discretization may be more efficient and accurate, especially for highly transient cases, where the time step limitation induced by the **cfl** condition is not too restrictive.

For the sake of readability, we establish this stability result in two steps: in Section 2, we address the case of a constant density flows, then we extend the proof to compressible flows in Section 3.

2 The incompressible case

Let Ω be split in control volumes $\bar\Omega = \cup_{K\in\mathcal{M}} \bar K$. We denote by $\mathcal{E}_{\rm int}$ the set of internal faces of the mesh, and by $\sigma = K|L$ the internal face separating control volumes K and L of $\mathcal{M}$.

In this section, we suppose that the density is constant, and, setting arbitrarily $\rho = 1$, the discrete finite volume convection operator which we study takes the following form:

$$\forall K \in \mathcal{M}, \qquad |K|\ C_K = \frac{|K|}{\delta t}\ (u_K - u_K^*) + \sum_{\sigma=K|L} F_{K,\sigma}\, u_\sigma^*, \tag{3}$$

where $F_{K,\sigma}$ stands for the discrete mass flux coming out from K through σ, the superscript * means that the quantity is taken at the beginning of the time step, and u_σ^* denotes the upwind (with respect to $F_{K,\sigma}$) approximation of u^* on σ, *i.e.* $u_\sigma^* = u_K^*$ if $F_{K,\sigma} \geq 0$ and $u_\sigma^* = u_L^*$ otherwise. We suppose that the evaluation of the (given) fluxes $F_{K,\sigma}$ from the field $\boldsymbol{q}$ is conservative, *i.e.* that, for an internal face $\sigma = K|L$, $F_{K,\sigma} = -F_{L,\sigma}$. Note that the fluxes through the external faces are implicitly set to zero (which is consistent with a given $\boldsymbol{q}$ supposed to vanish at the boundary). The incompressibility of the flow reads, at the discrete level:

$$\forall K \in \mathcal{M}, \qquad \sum_{\sigma=K|L} F_{K,\sigma} = 0. \tag{4}$$

Let us define the local **cfl** number associated to the mesh K by:

$$\mathbf{cfl}_K = \frac{\delta t}{|K|} \sum_{\sigma=K|L} \max\left(F_{K,\sigma}, 0\right) = \frac{\delta t}{|K|} \sum_{\sigma=K|L} -\min\left(F_{K,\sigma}, 0\right) \tag{5}$$

(the second equality resulting from the incompressibility relation (4)), and the global **cfl** number by:

$$\mathbf{cfl} = \max_{K \in \mathcal{M}} \mathbf{cfl}_K. \tag{6}$$

The stability of the convection operator defined by (3), *i.e.* the discrete analogue of (2) for a constant density flow, is stated in the following lemma.

LEMMA 2.1 Let **cfl** be defined by (5)-(6). For $K \in \mathcal{M}$, let C_K be defined by (3), and let Relation (4) hold. If $\mathbf{cfl} \leq 1$, then:

$$\sum_{K \in \mathcal{M}} |K|\ u_K\ C_K \geq \frac{1}{2\,\delta t} \sum_{K \in \mathcal{M}} |K|\ \big[(u_K)^2 - (u_K^*)^2\big].$$

Proof – We have $\sum_{K \in \mathcal{M}} |K|\ u_K\ C_K = T_1 + T_2$ with:

$$T_1 = \sum_{K \in \mathcal{M}} \frac{|K|}{\delta t}\ (u_K - u_K^*)\ u_K, \qquad T_2 = \sum_{K \in \mathcal{M}} u_K \sum_{\sigma = K|L} F_{K,\sigma}\, u_\sigma^*.$$

Using the identity $2a\,(a-b) = a^2 + (a-b)^2 - b^2$, valid for any real numbers a and b, we get for T_1:

$$T_1 = \frac{1}{2\,\delta t} \sum_{K \in \mathcal{M}} |K|\ \big[(u_K)^2 - (u_K^*)^2\big] + \frac{1}{2\,\delta t} \sum_{K \in \mathcal{M}} |K|\ (u_K - u_K^*)^2.$$

We now turn to T_2, which is split into $T_2 = T_{2,1} + T_{2,2}$ as follows:

$$T_{2,1} = \sum_{K \in \mathcal{M}} u_K^* \sum_{\sigma = K|L} F_{K,\sigma}\, u_\sigma^*, \qquad T_{2,2} = \sum_{K \in \mathcal{M}} (u_K - u_K^*) \sum_{\sigma = K|L} F_{K,\sigma}\, u_\sigma^*.$$

We now notice that, by definition of the upstream value u_σ^*, we have:

$$F_{K,\sigma}\ u_\sigma^* = |F_{K,\sigma}|\ \frac{u_K^* - u_L^*}{2} + F_{K,\sigma}\ \frac{u_K^* + u_L^*}{2},$$

so $T_{2,1}$ reads:

$$T_{2,1} = \sum_{K \in \mathcal{M}} u_K^* \sum_{\sigma = K|L} |F_{K,\sigma}|\ \frac{u_K^* - u_L^*}{2} + \sum_{K \in \mathcal{M}} u_K^* \sum_{\sigma = K|L} F_{K,\sigma}\ \frac{u_K^* + u_L^*}{2}. \tag{7}$$

First the incompressibility relation (4) then the conservativity yield for the second term:

$$\sum_{K \in \mathcal{M}} u_K^* \sum_{\sigma = K|L} F_{K,\sigma}\ \frac{u_K^* + u_L^*}{2} = \sum_{K \in \mathcal{M}} u_K^* \sum_{\sigma = K|L} F_{K,\sigma}\ \frac{u_L^*}{2} = 0.$$

Reordering now the first summation in (7), we get:

$$T_{2,1} = \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L} |F_{K,\sigma}|\ (u_K^* - u_L^*)^2.$$

Using once again Equation (4) to substract $F_{K,\sigma}\, u^*_K$ to all the fluxes at the faces of the mesh K, we have for $T_{2,2}$:

$$T_{2,2} = \sum_{K\in\mathcal{M}} (u_K - u^*_K) \sum_{\sigma=K|L,\ F_{K,\sigma}\leq 0} F_{K,\sigma}\,(u^*_L - u^*_K),$$

where the notation $\sum_{\sigma=K|L,\ F_{K,\sigma}\leq 0}$ means that the sum is restricted to the faces where the quantity $F_{K,\sigma}$ is non-positive. Reordering the summations, we get:

$$T_{2,2} = \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L,\ F_{K,\sigma}\leq 0} F_{K,\sigma}\ (u_K - u^*_K)\ (u^*_L - u^*_K),$$

where the above notation means that we perform the sum over each internal face σ, and we denote L the upwind control volume and K the downwind one. Using now the Cauchy-Schwarz and Young inequalities, we obtain:

$$T_{2,2} \geq -\frac{1}{2} \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} |F_{K,\sigma}|\ (u^*_L - u^*_K)^2 - \frac{1}{2} \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L,\ F_{K,\sigma}\leq 0} |F_{K,\sigma}|\ (u_K - u^*_K)^2.$$

The last summation reads:

$$\sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L,\ F_{K,\sigma}\leq 0} |F_{K,\sigma}|\ (u_K - u^*_K)^2 = \sum_{K\in\mathcal{M}} (u_K - u^*_K)^2 \sum_{\sigma=K|L} -\min(F_{K,\sigma}, 0).$$

Gathering the final expressions for T_1, $T_{2,1}$ and $T_{2,2}$, we obtain:

$$\begin{aligned}\sum_{K\in\mathcal{M}} |K|\ u_K\ C_K \ &\geq \frac{1}{2\,\delta t} \sum_{K\in\mathcal{M}} |K|\ \big[(u_K)^2 - (u^*_K)^2\big] \\ &+ \frac{1}{2} \sum_{K\in\mathcal{M}} (u_K - u^*_K)^2\ \Big[\frac{|K|}{\delta t} - \sum_{\sigma=K|L} -\min(F_{K,\sigma}, 0)\Big],\end{aligned}$$

which yields the conclusion. □

3 The compressible case

We now suppose that the flow is compressible, or, more precisely, that the density varies with time and space and that the momentum and the density are linked by the usual mass balance; the discrete mass balance now reads:

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}\ (\varrho_K - \varrho^*_K) + \sum_{\sigma=K|L} F_{K,\sigma} = 0, \tag{8}$$

and we study the following convection operator:

$$\forall K \in \mathcal{M}, \qquad |K|\ C_K = \frac{|K|}{\delta t}\ (\varrho_K u_K - \varrho^*_K u^*_K) + \sum_{\sigma=K|L} F_{K,\sigma}\, u^*_\sigma, \tag{9}$$

with the same definition for u^*_σ as in the previous section.

Let us define the local **cfl** number associated to the mesh K by:

$$\mathbf{cfl}_K = \frac{\delta t}{|K|\ \varrho_K} \sum_{\sigma=K|L} -\min\,(F_{K,\sigma}, 0), \tag{10}$$

the definition (6) of the global **cfl** number remaining unchanged.

The stability of the convection operator defined by (9), *i.e.* the discrete analogue of (2), is stated in the following lemma.

Lemma 3.1 Let **cfl** be defined by (10) and (6). For $K \in \mathcal{M}$, let C_K be defined by (9) and let Equation (8) hold. If $\mathbf{cfl} \leq 1$, then:

$$\sum_{K\in\mathcal{M}} |K|\ u_K\ C_K \geq \frac{1}{2\,\delta t} \sum_{K\in\mathcal{M}} |K|\ \big[\varrho_K\,(u_K)^2 - \varrho^*_K\,(u^*_K)^2\big].$$

Proof – We write $\displaystyle\sum_{K\in\mathcal{M}} |K|\ u_K\ C_K = T_1 + T_2$ with:

$$T_1 = \sum_{K\in\mathcal{M}} \frac{|K|}{\delta t}\ (\varrho_K\, u_K - \varrho^*_K\, u^*_K)\ u_K, \qquad T_2 = \sum_{K\in\mathcal{M}} u_K \sum_{\sigma=K|L} F_{K,\sigma}\, u^*_\sigma.$$

In T_1, let us first split $(\varrho_K\, u_K - \varrho^*_K\, u^*_K)\ u_K = \varrho_K\,(u_K - u^*_K)\,u_K + (\varrho_K - \varrho^*_K)\,u^*_K\,u_K$ and then use the identity $2a\,(a-b) = a^2 + (a-b)^2 - b^2$, valid for any real number a and b, to get:

$$\begin{aligned} T_1 \ &= \frac{1}{2\,\delta t} \sum_{K\in\mathcal{M}} |K|\ \varrho_K\ \big[(u_K)^2 - (u^*_K)^2\big] + \frac{1}{2\,\delta t} \sum_{K\in\mathcal{M}} |K|\ \varrho_K\ (u_K - u^*_K)^2 \\ &+ \underbrace{\frac{1}{\delta t} \sum_{K\in\mathcal{M}} |K|\ (\varrho_K - \varrho^*_K)\ u^*_K\ u_K\,.}_{T_{1,1}} \end{aligned}$$

We now write T_2 as:

$$T_2 = \sum_{K\in\mathcal{M}} u^*_K\ u_K \sum_{\sigma=K|L} F_{K,\sigma} + \sum_{K\in\mathcal{M}} u_K \sum_{\sigma=K|L} F_{K,\sigma}\ (u^*_\sigma - u^*_K).$$

By (8), the first summation in this relation is the opposite of $T_{1,1}$. As in the incompressible case, we split the second term in $T_{2,1} + T_{2,2} + T_{2,3}$ with:

$$\begin{aligned} T_{2,1} &= \sum_{K\in\mathcal{M}} u^*_K \sum_{\sigma=K|L} F_{K,\sigma}\ u^*_\sigma, \\ T_{2,2} &= \sum_{K\in\mathcal{M}} (u_K - u^*_K) \sum_{\sigma=K|L} F_{K,\sigma}\ (u^*_\sigma - u^*_K), \\ T_{2,3} &= -\sum_{K\in\mathcal{M}} (u^*_K)^2 \sum_{\sigma=K|L} F_{K,\sigma}. \end{aligned}$$

We remark that $T_{2,1}$ takes the same expression as in the incompressible case, so, by the same computation:

$$T_{2,1} = \frac{1}{2} \sum_{K \in \mathcal{M}} (u_K^*)^2 \sum_{\sigma=K|L} F_{K,\sigma} + \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\rm int},\ \sigma=K|L} |F_{K,\sigma}|\ (u_K^* - u_L^*)^2.$$

Using (8) for the first sum, we get:

$$T_{2,1} + T_{2,3} = \frac{1}{2} \sum_{K \in \mathcal{M}} \frac{|K|}{\delta t}\ (\varrho_K - \varrho_K^*)\ (u_K^*)^2 + \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\rm int},\ \sigma=K|L} |F_{K,\sigma}|\ (u_K^* - u_L^*)^2.$$

We now remark that the first of these terms combines with the first term of T_1 as follows:

$$\frac{1}{2\,\delta t} \sum_{K \in \mathcal{M}} |K|\ \Big[\varrho_K\ \big[(u_K)^2 - (u_K^*)^2\big] +\ (\varrho_K - \varrho_K^*)\ (u_K^*)^2\Big] = \frac{1}{2\,\delta t} \sum_{K \in \mathcal{M}} |K|\ \big[\varrho_K\ (u_K)^2 - \varrho_K^* (u_K^*)^2\big].$$

Gathering all terms, we conclude the proof by controlling the term $T_{2,2}$, which is the same as in the incompressible case, and can be absorbed by the same terms. □

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